Asymptotic Behaviour of Solutions of Nonlocal Parabolic Problems

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Abstract

In this thesis we deal with nonlinear nonlocal diffusion problems. The first chapter describes the motivation in studying this type of problems and the short overview of existing literature is given.

In Chapter 2 we address the issues of existence, uniqueness and the asymptotic behaviour of a solution $u = u(x,t)$ to the problem

$$\begin{cases} u_t - \nabla \cdot a(l(u(t)))|\nabla u|^{p-2}\nabla u = f & \text{in } \ \Omega \times (0,T), \\ \\ u = 0 & \text{on } \ \Gamma \times (0,T), \\ \\ u(\cdot,0) = u_0 & \text{in } \ \Omega, \end{cases} \tag{P1}$$

where $u(t) = u(\cdot,t)$ and $l(u(t))$ is defined by

$$l(u(t)) := \int_\Omega |\nabla u(x,t)|^p dx = \|\nabla u\|_p^p; \tag{L1}$$

$|\cdot|_p$ denotes the $L^p(\Omega)$-norm for $1 < p < +\infty$; Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 1$ with Lipschitz boundary Γ; $a(\xi) > 0$ is continuous and u_0, $f = f(x)$ are given data. In order to study the asymptotic behaviour of the solution of the problem (P1) we will first investigate the corresponding elliptic problem. This problem may have from one up to a continuum of solutions. Therefore, we show that the long time behaviour depends on the choice of the function a and the initial data u_0. Moreover, since the solutions of the stationary problem are also critical points of some energy functional, we classify its critical points. We prove that the solution of problem (P1) converges to a stationary solution, which is the global minimum of the energy functional, in case of uniqueness of such a stationary point. Furthermore, we also show local asymptotic stability of isolated local minima when $p \geq 2$.

In Chapter 3 we consider problem (P1) with

$$l(u(t)) := \int_\Omega g(x)u(x,t)dx, \tag{L2}$$

where g is a given function. We study the existence and uniqueness of solutions of problem (P1). Also in this case the corresponding stationary problem can have one, several or infinitely many solutions. We show that in case of uniqueness of the

solution of the stationary problem, the solution to the parabolic problem converges to the solution of the elliptic one. In this chapter we study as well in more details the problem (P1) in case when $f = \kappa g$, κ is some positive constant. To complete this chapter we give some remarks on the asymptotic behaviour of some problems related to (P1).

Finally in the last chapter we study existence of patterns for a reaction-diffusion system of population dynamics with nonlocal interaction. Namely, we study the following problem

$$\begin{cases} u_t = u_{xx} + u(1-u) - uv & \text{in } \Omega \times \mathbb{R}_+ \\ v_t = \lambda v_{xx} - \chi(u_x v)_x - \beta v + \delta \dfrac{\langle u, v\rangle}{\langle 1, v\rangle} v - \gamma \dfrac{uv}{1+\tau v} & \text{in } \Omega \times \mathbb{R}_+ \\ u_x = v_x = 0 & \text{in } \partial\Omega \times \mathbb{R}_+ \\ u = u_0, \ v = v_0 & \text{in } \Omega \times \{0\}\,. \end{cases} \tag{P2}$$

Here $\Omega \equiv (0,1)$, $\mathbb{R}_+ \equiv (0,\infty)$, $\partial\Omega \equiv \{0,1\}$, β, γ, δ, τ are positive constant coefficients, $\lambda > 0$ and $\chi \geq 0$ will be regarded as parameters, and

$$\langle u, v\rangle(t) := \int_0^1 u(x,t)v(x,t)\,dx\,, \quad \langle 1, v\rangle(t) := \int_0^1 v(x,t)\,dx \qquad (t \in \mathbb{R}_+)$$

for any measurable $u, v : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$. We address the system (P2) as a bifurcation problem (the bifurcation parameter being the diffusivity of one species λ), and investigate the possibility of patterns bifurcating out of a constant steady state solution via Turing destabilization. Additionally, we consider the local problem corresponding to the problem (P2), i.e. the problem (P2), where the nonlocal term $\delta \dfrac{\langle u, v\rangle}{\langle 1, v\rangle} v$ is replaced by δuv. It is shown that the nonlocal character of the interaction enhances the possibility that patterns exist with respect to the case of the companion problem with local interaction.

Zusammenfassung

In dieser Arbeit befassen wir uns mit nichtlinearen nichtlokalen Diffusionsproblemen. Das erste Kapitel erläutert die Motivation für die Untersuchung dieser Art von Problemen und es wird eine kurze Übersicht der vorhandenen Literatur gegeben.

Im zweiten Kapitel untersuchen wir Existenz, Eindeutigkeit und das asymptotische Verhalten einer Lösung $u = u(x,t)$ des Problems

$$\begin{cases} u_t - \nabla \cdot a(l(u(t)))|\nabla u|^{p-2}\nabla u = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \Gamma \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{P1}$$

wobei $u(t) = u(\cdot,t)$, und $l(u(t))$ durch

$$l(u(t)) := \int_\Omega |\nabla u(x,t)|^p dx = \|\nabla u\|_p^p, \tag{L1}$$

definiert ist; $|\cdot|_p$ bezeichnet die $L^p(\Omega)$-Norm für $1 < p < +\infty$ und Ω ist eine offene beschränkte Teilmenge des $\mathbb{R}^n$, $n \geq 1$ mit Lipschitz-Rand Γ; Ferner ist $a(\xi) > 0$ stetig und u_0, $f = f(x)$ sind gegeben. Um das asymptotische Verhalten der Lösung des Problems (P1) zu untersuchen, analysieren wir zuerst das entsprechende elliptische Problem. Dieses Problem kann eine bis zu einem Kontinuum von Lösungen haben. Deshalb zeigen wir, dass das Langzeitverhalten von der Wahl der Funktion a und der Anfangsbedingung u_0 abhängt. Ferner wissen wir, dass die Lösungen des stationären Problems auch die kristischen Punkte eines Energiefunktionales sind. Deshalb klassifizieren wir anschliessend diese Punkte. Wir beweissen, dass wenn das Energiefunktional ein eindeutig globales Minimum besitzt, die Lösung des Problems (P1) gegen diesen stationären Punkt konvergiert. Ausserdem zeigen wir die lokale Stabilität des isolierten lokalen Minimas im Fall $p \geq 2$.

Im dritten Kapitel betrachten wir das Problem (P1) mit

$$l(u(t)) := \int_\Omega g(x)u(x,t)dx, \tag{L2}$$

wobei g eine gegebene Funktion ist. Wir analysieren die Existenz und Eindeutigkeit der Lösung des Problems (P1). Auch in diesem Fall kann das entsprechende stationäre Problem eine, mehrere oder unendlich viele Lösungen haben. Wir zeigen,

dass wenn das stationäre Problem eine eindeutige Lösung besitzt, konvergiert die Lösung des parabolischen Problemes gegen die Lösung des elliptischen Problemes. In diesem Kapitel studieren wir auch detailierter das Problem (P1) mit $f = \kappa g$, wobei κ eine positive Konstante ist. Wir beenden das Kapitel mit ein paar Bemerkungken über das asymptotische Verhalten von Problemen, die mit (P1) zusammenhängend sind.

Wir untersuchen im letzten Kapitel die Existenz der Muster vom Reaktionsdiffusionssystem der Populationsdynamik mit nichtlokalen Interaktionen. Das heisst, dass wir das folgende Problem studieren:

$$\begin{cases} u_t = u_{xx} + u(1-u) - uv & \text{in } \Omega \times \mathbb{R}_+ \\ v_t = \lambda v_{xx} - \chi(u_x v)_x - \beta v + \delta \dfrac{\langle u, v\rangle}{\langle 1, v\rangle} v - \gamma \dfrac{uv}{1+\tau v} & \text{in } \Omega \times \mathbb{R}_+ \\ u_x = v_x = 0 & \text{in } \partial\Omega \times \mathbb{R}_+ \\ u = u_0,\ v = v_0 & \text{in } \Omega \times \{0\}\,. \end{cases} \tag{P2}$$

Wobei $\Omega \equiv (0,1)$, $\mathbb{R}_+ \equiv (0,\infty)$, $\partial\Omega \equiv \{0,1\}$ und β, γ, δ, τ positive konstante Koeffizienten sind; $\lambda > 0$ und $\chi \geq 0$ werden als Parameter betrachtet, und

$$\langle u, v\rangle(t) := \int_0^1 u(x,t)v(x,t)\,dx\,, \quad \langle 1, v\rangle(t) := \int_0^1 v(x,t)\,dx \qquad (t \in \mathbb{R}_+)$$

für beliebig messbare Funktionen $u, v : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$. Wir betrachten das System (P2) als Bifurkazionsproblem (der Bifurkationsparameter ist die Diffusionsfähigkeit einer Spezies λ), und analysieren die Möglichkeit von Mustern, welche sich aus einer konstanten Lösung durch Turing Destabilisierung verzweigen können. Zusätzlich untersuchen wir ein zu (P2) ähnliches Problem, der Unterschied ist, dass statt $\delta \dfrac{\langle u, v\rangle}{\langle 1, v\rangle} v$ ist neu δuv. Wir zeigen, dass der nichtlokale Charakter der Wechselwirkung die Möglichkeit, dass die Muster existieren, im Bezug auf das Begleitproblem mit lokaler Interaktion, erhöht.

Chapter 1

Introduction

1.1 Motivation

During the last decades a lot of attention has been devoted to nonlocal problems. Classical partial differential equations are local equations, which describes relations between the values of an unknown function at some point and its derivatives of different orders in an arbitrarily small neighborhood of this point. Instead, a nonlocal equation describes a nonlocal relation. That is in order to check whether a nonlocal equations holds at a point, information about the values of the function far from that point is needed. Most of the times, this is because the equation involves integral operators. This happens also in reality. Usually we do not have the information about studied objects and its features at every point, in reality the measurements are not made pointwise – but through some local average. This is one of the main reasons why we are interested in nonlocal problems and why such models became so widely investigated. Some interesting features of nonlocal problems and more motivation are described in [13], [16], [17], [12], [20] and in the references therein. In addition, nonlocal problems are challenging also from a mathematical point of view. Indeed, often they cannot be treated by classical methods for local PDEs, hence some new techniques have to be developed.

In particular, we shall concentrate on nonlocal parabolic problems associated with the p-Laplace operator. The study of p-Laplace operator is motivated by its various applications in physical and biological fields. For example, in Fluid Dynamics the shear stress $\overrightarrow{\tau}$ and the velocity gradient ∇u of the fluid are related in the following manner

$$\overrightarrow{\tau}(x) = r(x)|\nabla u|^{p-2}\nabla u,$$

where $p = 2$ (resp. $p < 2$, $p > 2$) if the fluid is Newtonian (resp. pseudoplastic or dilatant). Other applications of the p-Laplace operator arise in the study of flow through porous media ($p = \frac{3}{2}$), Nonlinear Elasticity ($p \geq 2$), Glaciology ($1 < p < \frac{4}{3}$) (see [3]). p-Laplacian appears also in the image processing, where the nonlinearity of the diffusion operator is used to eliminate the noise and many other fields (see for instance [5]). One can consider, in case of $p = 2$, the migration of a population (can be bacteria in a container [16]) or of a heat in a conductor of the phenomenon under the study.

Due to variety of applications of the p-Laplace operator and especially nonlocal problems, we combine both of this issues. Moreover, it is very natural to assume that the shear stress or the diffusion coefficient depend not on some local quantity, but nonlocal.

Another interesting issue is an interaction between two systems, one can consider not only the behaviour of one population, but also how do two populations coexist. Similar models can be described, for instance, by reaction-diffusion systems. Such systems with nonlocal interactions arise in a variety of applications, particularly in models of mathematical biology (*e.g.*, see [11, 27, 29, 32, 33, 46, 47] and references therein), the motivation for their introduction depending on the context. For instance, in epidemiological models it is conceivable that the presence of infectives at some point influences some surrounding region as far as the spread of epidemics is concerned, whereas in population dynamics one can think of a population whose individuals communicate by chemical means, or compete for some resource which can rapidly redistribute itself, *e.g.* by convection. Nonlocal terms in equations modelling population dynamics can also arise by very different factors (*e.g.*, see [8, 25, 44]), or derive by some limiting procedure (as in the "shadow system" associated to some reaction-diffusion system with local interaction [34, 35, 41, 42]).

1.2 Overview of the literature

Local case

The simplest model problem of parabolic type involving p-Laplacian is the next problem

$$\begin{cases} u_t - \nabla \cdot |\nabla u|^{p-2}\nabla u = f & \text{in } \ \Omega \times (0,T), \\ u = 0 & \text{on } \ \Gamma \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \ \Omega, \end{cases} \tag{1.2.1}$$

where the exponent $p \in (1,+\infty)$, Ω is a bounded domain in $\mathbb{R}^N$, Γ its boundary and f – external source, u_0 – initial condition are given. There is an extensive literature devoted to this problem, we limit ourselves just by referring the monographs [38], [26] and the paper [6], where the questions of existence, uniqueness and regularity of the solution are discussed. In particular, in those works the existence and uniqueness of the solution is obtained via monotonicity and compactness methods.

Corresponding to (1.2.1) stationary problem, i.e. the following elliptic problem

$$\begin{cases} -\nabla \cdot |\nabla u|^{p-2}\nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1.2.2}$$

is also well understood. It is known that the solution to this problem is unique and can be found by the variational method as a minimizer of the energy functional

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega fudx, \tag{1.2.3}$$

(see for instance [14]).

One more very significant issue which was addressed to the problem (1.2.1) is its dynamics, namely the behaviour of the solution when time goes to infinity. We know that the extinction in a finite time happens when $1 < p < 2$ and $f = 0$ (see [26]) or in more general case we have that

$$u(t) \to u_\infty \text{ as } t \to +\infty \text{ in } W_0^{1,p}(\Omega), \tag{1.2.4}$$

where $u(t) = u(\cdot, t)$ is a solution to the parabolic problem (1.2.1) and u_∞ denotes a unique solution to the elliptic problem (1.2.2) (see [39]). More general stability issues for a local case with $f = f(x,u)$ were considered in [7].

Nonlocal elliptic problem

The elliptic problems with our type of nonlocality have been studied in [24], [23]. More precisely, the problem

$$\begin{cases} -M(\|\nabla u\|_p^p)^{p-1}\nabla \cdot |\nabla u|^{p-2}\nabla u = f(x,u) & \text{in } \Omega, \\ \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1.2.5}$$

was under the study. The existence of nonnegative solution was obtained. In the first paper this problem was considered as a limit problem of some perturbed problem and in the second note Mountain Pass Lemma was used.

Nonlocal Laplace operator

When $p = 2$, p-Laplacian is just a usual Laplace operator. Problem (P1) for $p = 2$ was studied in the papers [16], [17] and in the book [13]. This results were also extended in [20], [12] for a more general second order linear operator with the nonlocal quantity present in a diffusion coefficient and of the type of (L2). In these works the main method to study the asymptotic behaviour of a solution relied on the special structure of the nonlocal problems and known theory of the dynamical systems (see [13, 48]). One could reduce solving of the stationary problem to solving of some particular equation in $\mathbb{R}$. For instance, in case of Laplace operator, we know that the stationary problem has as many solutions as the equation

$$a(\mu)\mu = l(\varphi),$$

where φ is a solution to (1.2.2) and l is given by (L2). Therefore, for a different function a the stationary problem can have one, several or infinitely many solutions. Hence, the question of the asymptotic behaviour becomes not trivial. In the above mentioned papers the authors showed that the solution to problem (P1) converges to the solution of the corresponding elliptic problem in $L^2(\Omega)$. The results which are obtained in these papers are local, except for the case of the stationary problem having a unique solution. Furthermore, recently in [9] the authors studied existence and regularity of pullback attractors (both in $L^2(\Omega)$ and in $H_0^1(\Omega)$) of the solution of problem (P1), where f is given by $f = g(u) + h(t)$.

Some methods to study problem (P1) with l given by (L1) for $p = 2$ were given in [21] and [22] for a weak and strong solutions respectively. Again the particular structure of the stationary problem was used, this time combined with the existence of Lyapunov function. Also in this case the stationary problem can have from one up to a infinitely many of solutions. Moreover, it was shown that the stationary solutions can be found as critical points of the energy functional given by

$$E(u) = \frac{1}{2}A\left(\int_\Omega |\nabla u|^2 dx\right) - \int_\Omega f u dx,$$

where

$$A(z) = \int_0^z a(s)ds.$$

The convergence results were obtained in $H_0^1(\Omega)$ and $H^2(\Omega)$ in [21] and [22] respectively and were local.

Chapter 2

Nonlocal p-Laplace equations depending on the L^p norm of the gradient

In this chapter we consider the problem of finding $u = u(x,t)$ solution to

$$\begin{cases} u_t - \nabla \cdot a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \Gamma \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{2.0.1}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 1$ with Lipschitz boundary Γ. We assume

$$a \text{ is continuous, } a(\xi) > 0, \ \forall \xi \in \mathbb{R}. \tag{2.0.2}$$

By $|\cdot|_p$, we denote the $L^p(\Omega)$-norm, $1 < p < +\infty$ and we assume

$$f = f(x) \in W^{-1,q}(\Omega) := (W_0^{1,p}(\Omega))^*, \ u_0 \in W_0^{1,p}(\Omega) \cap L^2(\Omega), \ \tfrac{1}{p} + \tfrac{1}{q} = 1. \tag{2.0.3}$$

In what follows we will denote by $|\cdot|_{-1,q}$ – a norm in $W^{-1,q}(\Omega)$. For notions on Sobolev spaces, we refer to [13], [28], [38].

2.1 Existence

Theorem 2.1.1. *Let the assumptions above hold and assume that there exist two constants λ, Λ such that*

$$0 < \lambda \leq a(\xi) \leq \Lambda, \quad \forall \xi \in \mathbb{R}. \tag{2.1.1}$$

Then, if $f \in L^q(\Omega) \subset W^{-1,q}(\Omega)$ for every $T > 0$ there exists u solution to

$$\begin{cases} u \in L^p(0,T; W_0^{1,p}(\Omega)) \cap C([0,T]; L^r(\Omega)), \ r = \min\{2,p\}, \\ u_t \in L^q(0,T; W^{-1,q}(\Omega)), \quad u(\cdot,0) = u_0, \\ \langle u_t, v\rangle + \int_\Omega a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u \nabla v dx = \langle f, v\rangle \\ \forall v \in W_0^{1,p}(\Omega) \ in \ \mathcal{D}'(0,T), \end{cases} \tag{2.1.2}$$

where $\langle \cdot,\cdot\rangle$ denotes the pairing between $W^{-1,q}(\Omega)$ and $W_0^{1,p}(\Omega)$, $u(t) = u(\cdot,t)$, $\mathcal{D}'(0,T)$ is the space of distributions on $(0,T)$.

Proof. We consider $\lambda_1, \ldots, \lambda_n, \ldots$ a basis in $W_0^{1,p}(\Omega) \cap L^2(\Omega)$ consisting of eigenvalues of the problem

$$\lambda_i \in H_0^s(\Omega), \ (\lambda_i, v)_{H_0^s(\Omega)} = \mu_i(\lambda_i, v)_{L^2(\Omega)} \quad \forall v \in H_0^s(\Omega),$$

where s is chosen large enough in such a way that $H_0^s(\Omega) \subset W_0^{1,p}(\Omega)$ continuously. We will suppose the λ_i's orthonormal in $L^2(\Omega)$. If $u_0 = \sum_i \beta_i\lambda_i$ consider

$$u_n(t) = \sum_{i=1}^n \gamma_i(t)\lambda_i$$

solution to

$$\begin{cases} \int_\Omega u_n' v dx + a(\|\nabla u_n\|_p^p)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v dx = \langle f, v\rangle, \\ u_n(0) = \sum_{i=1}^n \beta_i\lambda_i, \qquad \forall v \in [\lambda_1, \ldots, \lambda_n], \end{cases} \tag{2.1.3}$$

where $[\lambda_1, \ldots, \lambda_n]$ is the space spanned by $\lambda_1, \ldots, \lambda_n$. Taking $v = \lambda_j$ and using the fact that the λ_i's are orthonormal, we see that (2.1.3) is equivalent to the Cauchy problem

$$\gamma_j'(t) = -a\Big(\Big\|\sum_{i=1}^n \gamma_i(t)\nabla\lambda_i\Big\|_p^p\Big)\int_\Omega\Big|\sum_{i=1}^n \gamma_i(t)\nabla\lambda_i\Big|^{p-2}\sum_{i=1}^n \gamma_i(t)\nabla\lambda_i\nabla\lambda_j dx + \langle f, \lambda_j\rangle,$$
$$\gamma_j(0) = \beta_j, \quad \forall j = 1, \ldots n. \tag{2.1.4}$$

Since the right hand side of the first equation above is continuous in γ_i this Cauchy problem possesses a solution. Moreover, using the formulation (2.1.3) and taking $v = u_n$, we see that

$$\int_\Omega u_n' u_n dx + a(\|\nabla u_n\|_p^p)\int_\Omega |\nabla u_n|^p dx = \langle f, u_n\rangle,$$

which implies using (2.1.1), Poincaré's and Young's inequalities

$$\frac{1}{2}\frac{d}{dt}|u_n|_2^2 + \lambda \int_\Omega |\nabla u_n|^p dx \le C|f|_q \|\nabla u_n\|_p \le \varepsilon \|\nabla u_n\|_p^p + C_\varepsilon |f|_q^q.$$

Choosing for instance $\varepsilon = \frac{\lambda}{2}$, we arrive to

$$\frac{1}{2}\frac{d}{dt}|u_n|_2^2 + \frac{\lambda}{2} \int_\Omega |\nabla u_n|^p dx \le C_\varepsilon |f|_q^q.$$

After an integration in t this leads to

$$\frac{1}{2}|u_n(t)|_2^2 + \frac{\lambda}{2}\int_0^t \int_\Omega |\nabla u_n|^p dxdt \le C_\varepsilon \int_0^t |f|_q^q dt + \frac{1}{2}|u_n(0)|_2^2. \tag{2.1.5}$$

In particular, we see that $|u_n(t)|_2$ remains bounded in time and thus, the solution to (2.1.3) or (2.1.4) is global in time ($|\cdot|_2$ is just a norm in $[\lambda_1, \dots, \lambda_n]$, where all the norms are equivalent).

Remark that $\|\nabla u_n\|_p$ remains bounded in time uniformly. To see that taking $v = u_n'$ in (2.1.3), we get

$$\int_\Omega u_n'^2 dx + a(\|\nabla u_n\|_p^p) \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla u_n' dx = \langle f, u_n' \rangle. \tag{2.1.6}$$

Introducing

$$E(u) = \frac{1}{p} A\Big(\int_\Omega |\nabla u|^p dx\Big) - \langle f, u \rangle \tag{2.1.7}$$

with

$$A(z) = \int_0^z a(s) ds, \tag{2.1.8}$$

we see that (2.1.6) can be written

$$\partial_t E(u_n) = -\int_\Omega u_n'^2 dx \le 0. \tag{2.1.9}$$

Thus $E(u_n)$ decreases in time and is bounded from above for every t. The bound for $\|\nabla u_n\|_p$ follows then from the estimate

$$E(u_n) \ge \frac{\lambda}{p}\|\nabla u_n\|_p^p - C|f|_q \|\nabla u_n\|_p. \tag{2.1.10}$$

From (2.1.5), (2.1.10), we deduce that

$$u_n \text{ is bounded in } L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(0,T; L^2(\Omega) \cap W_0^{1,p}(\Omega)).$$

Furthermore, from the first equation in (2.1.3) we derive easily if P_n denotes the orthogonal projection from $L^2(\Omega)$ onto $[\lambda_1, \dots, \lambda_n]$ for every $v \in H_0^s(\Omega)$

$$\int_\Omega u_n' v dx = \int_\Omega u_n' P_n(v) dx \le \Big(\Lambda\Big(\int_\Omega |\nabla u_n|^{(p-1)q} dx\Big)^{\frac{1}{q}} + C|f|_q\Big) \|\nabla(P_n v)\|_p$$

$$\leq C\Big(\Lambda\|\nabla u_n\|_p^{p-1} + |f|_q\Big)|P_n v|_{H_0^s(\Omega)} \leq C\Big(\Lambda\|\nabla u_n\|_p^{p-1} + |f|_q\Big)|v|_{H_0^s(\Omega)}$$

(C above denotes various constants independent of n). Thus

$$|u_n'|_{H^{-s}(\Omega)} \leq C\Big(\Lambda\|\nabla u_n\|_p^{p-1} + C|f|_q\Big),$$

and u_n' is bounded in $L^q(0,T;H^{-s}(\Omega))$. Since $W_0^{1,p}(\Omega) \subset L^p(\Omega) \subset H^{-s}(\Omega)$ and since the first embedding is compact (see [28]), we have that

$$W := \{v \in L^p(0,T;W_0^{1,p}(\Omega)),\ v' \in L^q(0,T;H^{-s}(\Omega))\}$$

is compactly embedded in $L^p(0,T;L^p(\Omega)) = L^p(Q_T)$, $Q_T = \Omega \times (0,T)$. Suppose first $p \geq 2$. Then we can find a subsequence of n such that

$$u_n \rightharpoonup u \text{ in } L^p(0,T;W_0^{1,p}(\Omega)),$$

$$u_n \to u \text{ in } L^p(0,T;L^p(\Omega)),$$

$$u_n' \rightharpoonup u' \text{ in } L^q(0,T;H^{-s}(\Omega)),$$

$$\frac{1}{a(\|\nabla u_n\|_p^p)} \rightharpoonup a_\infty \text{ in } L^\infty(0,T) \text{ - weak*},$$

$$u_n(T) \rightharpoonup u(T) \text{ in } L^2(\Omega),$$

$$\nabla \cdot |\nabla u_n|^{p-2}\nabla u_n \rightharpoonup \chi \text{ in } L^q(0,T;W^{-1,q}(\Omega)).$$

In fact,

$$u_n' \in L^2(0,T;L^2(\Omega)) = L^2(Q_T). \tag{2.1.11}$$

Indeed, integrating (2.1.9) from 0 to T, we derive

$$\int_0^T \int_\Omega |u_n'|^2 dx = E(u_n(0)) - E(u_n(T)). \tag{2.1.12}$$

Using the Young inequality in (2.1.10), we get

$$E(u) \geq \frac{\lambda}{p}\|\nabla u\|_p^p - \frac{C|f|_q^q}{\lambda^{\frac{q}{p}} q} - \frac{\lambda}{p}\|\nabla u\|_p^p = -\frac{1}{q}\Big(\frac{C|f|_q}{\lambda^{\frac{1}{p}}}\Big)^q,$$

hence, $E(u_n)$ is bounded from below independently of n. Thus, from (2.1.12), we obtain (2.1.11). Therefore, in the case $1 < p < 2$ one has with bounds independent of n

$$u_n \in L^p(0,T;W_0^{1,p}(\Omega)),\ u_n' \in L^2(0,T;L^2(\Omega)) \subset L^2(0,T;L^p(\Omega)).$$

Hence, by the Aubin-Lions Lemma the embedding of $\{v \in L^p(0,T;W_0^{1,p}(\Omega)),\ v' \in L^2(0,T;L^p(\Omega))\}$ in $L^p(0,T;L^p(\Omega))$ is compact and one can extract a sequence of u_n satisfying the same convergences as above with $u_n' \rightharpoonup u'$ in $L^2(0,T;L^2(\Omega))$. The fact that $u \in C([0,T], L^r(\Omega))$ with $r = \min\{2,p\}$ follows by the standard argument (see [38]). By rescaling the time in the following way, setting

$$\alpha(t) = \int_0^t a(\|\nabla u(\cdot,s)\|_p^p)ds, \tag{2.1.13}$$

we reduce solving the problem (2.0.1) to solving the problem (see [21]):

$$\begin{cases} w_t - \nabla \cdot |\nabla w|^{p-2}\nabla w = \dfrac{f}{a(\|\nabla w\|_p^p)} & \text{in } \Omega \times (0, \alpha(T)), \\ w = 0 & \text{on } \Gamma \times (0, \alpha(T)), \\ w(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases} \tag{2.1.14}$$

where $w(x, \alpha(t)) = u(x,t)$. Replacing in (2.1.13) u by u_n, we can also write the first equation of (2.1.3) as

$$\int_\Omega u_n' v dx + \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v dx = \frac{\langle f, v\rangle}{a(\|\nabla u_n\|_p^p)}. \tag{2.1.15}$$

Now passing to the limit in (2.1.15) one has in the distributional sense in Q_T

$$u_t - \chi = a_\infty f \tag{2.1.16}$$

(therefore, $u_t \in L^q(0,T; W^{-1,q}(\Omega))$). Taking $v = u_n$ in (2.1.15), we obtain

$$\frac{1}{2}\frac{d}{dt}|u_n|_2^2 + \int_\Omega |\nabla u_n|^p dx = \frac{\langle f, u_n\rangle}{a(\|\nabla u_n\|_p^p)}$$

and by integration on $(0,T)$, we get

$$\int_{Q_T} |\nabla u_n|^p dxdt = \int_0^T \frac{\langle f, u_n\rangle}{a(\|\nabla u_n\|_p^p)} dt + \frac{|u_n(0)|_2^2}{2} - \frac{|u_n(T)|_2^2}{2}. \tag{2.1.17}$$

Since $u_n \to u$ in $L^p(Q_T)$, $\frac{f}{a(\|\nabla u_n\|_p^p)} \rightharpoonup a_\infty f$ in $L^q(Q_T)$ and using the fact that $\varliminf_{n\to\infty} |u_n(T)|_2^2 \geq |u(T)|_2^2$ from (2.1.17), we get

$$\varlimsup_{n\to\infty} \int_{Q_T} |\nabla u_n|^p dxdt \leq \int_0^T a_\infty \langle f, u\rangle dt + \frac{|u_0|_2^2}{2} - \frac{|u(T)|_2^2}{2}. \tag{2.1.18}$$

Thus, from the inequality

$$\int_{Q_T} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla v|^{p-2}\nabla v) \cdot \nabla(u_n - v) dxdt \geq 0,$$

we derive by taking the $\overline{\lim}$ for any $v \in L^p(0,T; W_0^{1,p}(\Omega))$

$$\int_0^T a_\infty \langle f, u\rangle dt + \frac{|u_0|_2^2}{2} - \frac{|u(T)|_2^2}{2} + \int_0^T \langle \chi, v\rangle dt - \int_{Q_T} |\nabla v|^{p-2}\nabla v \cdot \nabla(u - v) dxdt \geq 0. \tag{2.1.19}$$

By integrating (2.1.16) after having multiplied by u, we get

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 - \langle \chi, u\rangle = a_\infty \langle f, u\rangle$$

and integrating over $(0,T)$, we obtain

$$-\int_0^T \langle \chi, u\rangle dt = \int_0^T a_\infty \langle f, u\rangle dt + \frac{|u_0|_2^2}{2} - \frac{|u(T)|_2^2}{2}. \tag{2.1.20}$$

Thus, combining (2.1.19), (2.1.20), we have

$$\int_0^T \langle -\chi + \nabla\cdot|\nabla v|^{p-2}\nabla v, u - v\rangle dt \geq 0 \quad \forall v \in L^p(0,T;W_0^{1,p}(\Omega)).$$

Taking $v = u - \delta w,\ \delta > 0$, we see

$$\int_0^T \langle -\chi + \nabla\cdot|\nabla(u-\delta w)|^{p-2}\nabla(u-\delta w), w\rangle dt \geq 0 \ \forall w \in L^p(0,T;W_0^{1,p}(\Omega)).$$

Letting $\delta \to 0$, we get easily

$$\int_0^T \langle -\chi + \nabla\cdot|\nabla u|^{p-2}\nabla u, w\rangle dt = 0 \quad \forall w \in L^p(0,T;W_0^{1,p}(\Omega))$$

and the equation (2.1.16) reads

$$u_t - \nabla\cdot|\nabla u|^{p-2}\nabla u = a_\infty f.$$

Going back to (2.1.18), (2.1.20), we derive

$$\overline{\lim_{n\to\infty}}\int_{Q_T}|\nabla u_n|^p dxdt \leq \int_{Q_T}|\nabla u|^p dxdt\Big(\leq \underline{\lim_{n\to\infty}}\int_{Q_T}|\nabla u_n|^p dxdt\Big)$$

and $\nabla u_n \to \nabla u$ in $L^p(Q_T)$ strongly. Up to a subsequence, we have

$$\int_\Omega |\nabla(u_n - u)|_p^p dxdt \to 0 \ \text{ a.e. } t,$$

i.e. this implies $\|\nabla u_n\|_p^p \to \|\nabla u\|_p^p$ a.e. t and then $\frac{1}{a(\|\nabla u_n\|_p^p)} \to \frac{1}{a(\|\nabla u\|_p^p)}$ a.e. t, since the sequence is bounded this convergence take also place in any $L^p(0,T)$ and $a_\infty = \frac{1}{a(\|\nabla u\|_p^p)}$, which completes the proof. □

2.2 Uniqueness

Theorem 2.2.1. *If in addition to the assumptions of Theorem* 2.1.1 *for some L it holds that*

$$|a(\xi) - a(\xi')| \leq L|\xi - \xi'| \quad \forall \xi, \xi' \in \mathbb{R} \tag{2.2.1}$$

and $f \in L^2(\Omega)$, *then the solution to* (2.1.2) *is unique.*

Proof. Let u_1, u_2 be two weak solutions to

$$\begin{cases} u \in L^p(0,T;W_0^{1,p}(\Omega)), \quad u_t \in L^q(0,T;W^{-1,q}(\Omega)), \\ u_t - \nabla\cdot|\nabla u|^{p-2}\nabla u = \dfrac{f}{a(\|\nabla u\|_p^p)}. \end{cases} \tag{2.2.2}$$

By subtraction, we obtain

$$(u_1-u_2)_t - \nabla\cdot\big(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\big) = \Big(\frac{1}{a(\|\nabla u_1\|_p^p)} - \frac{1}{a(\|\nabla u_2\|_p^p)}\Big)f.$$

Multiplying by u_1-u_2, integrating over Ω and using (2.1.1), (2.2.1), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2 &+ \int_\Omega \big(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\big)\cdot\nabla(u_1-u_2)dx \\ &\le \frac{L}{\lambda^2}\big|\|\nabla u_1\|_p^p - \|\nabla u_2\|_p^p\big|\left|\int_\Omega f(u_1-u_2)dx\right|. \end{aligned} \tag{2.2.3}$$

From Lemma A.3 and the Hölder inequality, we derive

$$\begin{aligned} &\big|\|\nabla u_1\|_p^p - \|\nabla u_2\|_p^p\big| = \Big|\int_\Omega(|\nabla u_1|^p - |\nabla u_2|^p)dx\Big| \le \int_\Omega\Big||\nabla u_1|^p - |\nabla u_2|^p\Big|dx \\ &\le p\int_\Omega\big(|\nabla u_1|+|\nabla u_2|\big)^{p-1}|\nabla(u_1-u_2)|dx \\ &= p\int_\Omega\big(|\nabla u_1|+|\nabla u_2|\big)^{\frac{p}{2}}\big(|\nabla u_1|+|\nabla u_2|\big)^{\frac{p}{2}-1}|\nabla(u_1-u_2)|dx \\ &\le p\Big(\int_\Omega\big(|\nabla u_1|+|\nabla u_2|\big)^p dx\Big)^{\frac12}\Big(\int_\Omega\big(|\nabla u_1|+|\nabla u_2|\big)^{p-2}|\nabla(u_1-u_2)|^2dx\Big)^{\frac12}. \end{aligned} \tag{2.2.4}$$

From Lemma A.1, we obtain

$$\begin{aligned} &\int_\Omega(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)\cdot\nabla(u_1-u_2)dx \\ &\qquad\ge c_p\int_\Omega\big(|\nabla u_1|+|\nabla u_2|\big)^{p-2}|\nabla(u_1-u_2)|^2dx. \end{aligned}$$

Combining (2.2.3) and the two inequalities above, leads to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2 + c_p\int_\Omega\big(|\nabla u_1|+|\nabla u_2|\big)^{p-2}|\nabla(u_1-u_2)|^2dx \\ &\le \frac{Lp}{\lambda^2}\Big|\int_\Omega f(u_1-u_2)dx\Big|\Big(\int_\Omega\big(|\nabla u_1|+|\nabla u_2|\big)^p dx\Big)^{\frac12} \\ &\times\Big(\int_\Omega\big(|\nabla u_1|+|\nabla u_2|\big)^{p-2}|\nabla(u_1-u_2)|^2dx\Big)^{\frac12} \\ &\le \frac{c_p}{2}\int_\Omega\big(|\nabla u_1|+|\nabla u_2|\big)^{p-2}|\nabla(u_1-u_2)|^2dx + C(t)\int_\Omega|u_1-u_2|^2dx. \end{aligned}$$

(In the last inequality above, we use Young's inequality. Note that $C \in L^1(0,T)$). Therefore, we have

$$\frac{1}{2}\frac{d}{dt}|u_1 - u_2|_2^2 \le C(t)\int_\Omega |u_1 - u_2|^2 dx.$$

The uniqueness follows then from Gronwall's inequality. □

Theorem 2.2.2. *Let the assumptions* (2.0.2), (2.0.3) *hold and if in addition the function a is such that*

$$s \mapsto a(s^p)s^{p-1} \text{ is nondeacreasing,} \tag{2.2.5}$$

then the solution to (2.1.2) *is unique.*

Proof. Let u_1, u_2 be two solutions to (2.1.2), then taking $v = u_1 - u_2$ and by subtraction, one has

$$\frac{1}{2}\frac{d}{dt}|u_1 - u_2|_2^2 + \int_\Omega \Big(a(\|\nabla u_1\|_p^p)|\nabla u_1|^{p-2}\nabla u_1 - a(\|\nabla u_2\|_p^p)|\nabla u_2|^{p-2}\nabla u_2\Big)\nabla(u_1 - u_2)dx = 0. \tag{2.2.6}$$

By expanding the integral term I, one gets

$$I = \int_\Omega \Big(a(\|\nabla u_1\|_p^p)|\nabla u_1|^p - a(\|\nabla u_1\|)|\nabla u_1|^{p-2}\nabla u_1\nabla u_2 + a(\|\nabla u_2\|_p^p)|\nabla u_2|^p - a(\|\nabla u_2\|)|\nabla u_2|^{p-2}\nabla u_2\nabla u_1\Big)dx.$$

Recall that $a(\|\nabla u_i\|_p^p)$, $i = 1, 2$ are independent of x and can be pulled out of the integrals. Using Hölder's inequality, we see

$$\int_\Omega |\nabla u_1|^{p-2}\nabla u_1\nabla u_2 dx \le \|\nabla u_2\|_p\|\nabla u_1\|_p^{p-1},$$

$$\int_\Omega |\nabla u_2|^{p-2}\nabla u_2\nabla u_1 dx \le \|\nabla u_1\|_p\|\nabla u_2\|_p^{p-1}.$$

Then using (2.2.5), we get

$$\begin{aligned} I &\ge a(\|\nabla u_1\|_p^p)\Big(\|\nabla u_1\|_p^p - \|\nabla u_1\|_p^{p-1}\|\nabla u_2\|_p\Big) \\ &\quad + a(\|\nabla u_2\|_p^p)\Big(\|\nabla u_2\|_p^p - \|\nabla u_2\|_p^{p-1}\|\nabla u_1\|_p\Big) \\ &= \Big(a(\|\nabla u_1\|_p^p)\|\nabla u_1\|_p^{p-1} - a(\|\nabla u_2\|_p^p)\|\nabla u_2\|_p^{p-1}\Big)(\|\nabla u_1\|_p - \|\nabla u_2\|_p) \ge 0. \end{aligned}$$

Hence, (2.2.6) implies $\frac{d}{dt}|u_1 - u_2|_2^2 \le 0$, therefore, the result follows. □

Remark 2.2.1. Note that (2.2.5) holds in particular for a nondecreasing.

2.3 The stationary problem

In this section, we consider the associated stationary problem to the problem (2.0.1), that is the following problem

$$\begin{cases} -\nabla \cdot a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{2.3.1}$$

We will assume here that $f \in W^{-1,q}(\Omega)$. In a weak form u is a weak solution to

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u \nabla v dx = \langle f, v\rangle \quad \forall v \in W_0^{1,p}(\Omega). \end{cases} \tag{2.3.2}$$

In order to solve the stationary problem, we introduce φ the solution to

$$\begin{cases} \varphi \in W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega |\nabla \varphi|^{p-2}\nabla \varphi \nabla v dx = \langle f, v\rangle \quad \forall v \in W_0^{1,p}(\Omega). \end{cases} \tag{2.3.3}$$

It is known that for $f \in W^{-1,q}(\Omega)$ (2.3.3) admits a unique solution [14].

Theorem 2.3.1. *Suppose that* (2.0.2) *holds,* $1 < p < +\infty$. *Then for* $f \in W^{-1,q}(\Omega)$, *the mapping* $u \mapsto \|\nabla u\|_p^p$ *is one-to-one mapping from the set of solutions to* (2.3.2) *onto the set of solutions in* $\mathbb{R}$ *of the equation*

$$a(\mu)^{\frac{p}{p-1}}\mu = \|\nabla \varphi\|_p^p. \tag{2.3.4}$$

Proof. Let u be a solution to the stationary problem, then

$$\begin{aligned} \int_\Omega a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u \nabla v dx &= \langle f, v\rangle \\ &= \int_\Omega |\nabla \varphi|^{p-2}\nabla \varphi \nabla v dx \quad \forall v \in W_0^{1,p}(\Omega), \end{aligned} \tag{2.3.5}$$

which implies

$$a(\|\nabla u\|_p^p)^{\frac{1}{p-1}}u = \varphi, \tag{2.3.6}$$

from where follows

$$a(\|\nabla u\|_p^p)^{\frac{p}{p-1}}\|\nabla u\|_p^p = \|\nabla \varphi\|_p^p. \tag{2.3.7}$$

Hence, $\|\nabla u\|_p^p$ is a solution to (2.3.4).

Let now μ be a solution to (2.3.4), u denotes the solution to

$$u \in W_0^{1,p}(\Omega), \quad \int_\Omega a(\mu)|\nabla u|^{p-2}\nabla u \nabla v dx = \langle f, v\rangle \quad \forall v \in W_0^{1,p}(\Omega), \tag{2.3.8}$$

then $a(\mu)^{\frac{1}{p-1}} u = \varphi$. Therefore, we get

$$a(\mu)^{\frac{p}{p-1}} \|\nabla u\|_p^p = \|\nabla \varphi\|_p^p = a(\mu)^{\frac{p}{p-1}} \mu \ \Rightarrow\ \|\nabla u\|_p^p = \mu$$

and u is a solution to (2.3.2). Now to show the injectivity, we have

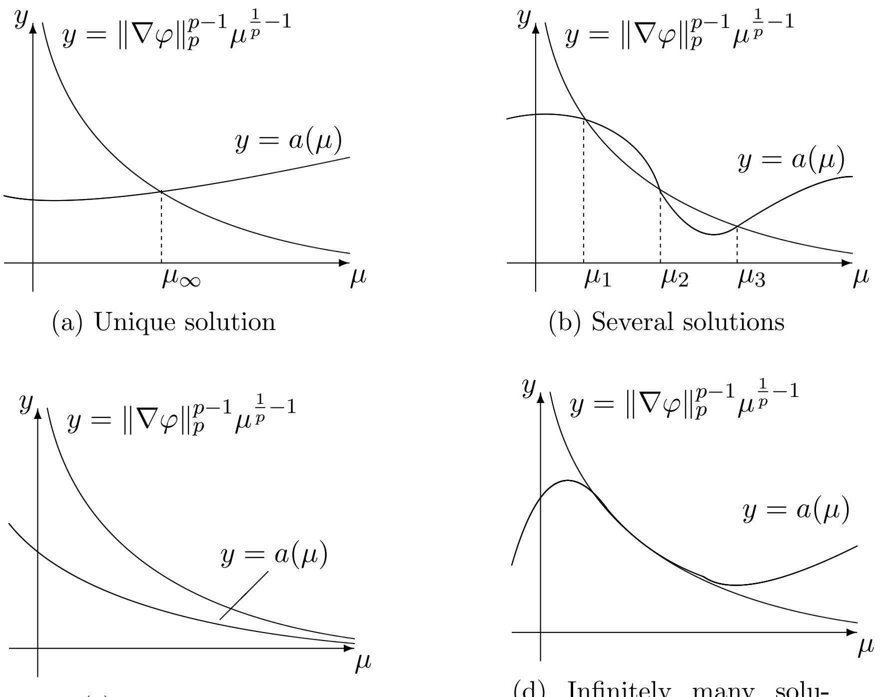
$$\|\nabla u_1\|_p^p = \|\nabla u_2\|_p^p \ \Rightarrow\ a(\|\nabla u_1\|_p^p) = a(\|\nabla u_2\|_p^p) \ \Rightarrow\ u_1 = u_2,$$

due to the uniqueness of the solution of (2.3.8). □

Remark 2.3.1. The stationary points are determined by the solutions to

$$a(\mu) = \|\nabla \varphi\|_p^{p-1} \mu^{\frac{1}{p}-1}. \tag{2.3.9}$$

Thus it can happen that there is one solution, several, infinitely many solutions or no solution (just in case where a is not bounded away from 0). It depends on the function a, see Figure 2.3.1. In the case where (2.2.5) holds the set of stationary points is an interval which is reduced to a point when $a(s^p)s^{p-1}$ is increasing.



(a) Unique solution (b) Several solutions

(c) No solution (d) Infinitely many solutions

Figure 2.3.1

The solutions of the problem (2.3.2) can be also found as critical points of the energy $E(u)$, defined by (2.1.7), (2.1.8) and

$$E'(u) = -\nabla \cdot a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u - f. \tag{2.3.10}$$

If u_∞ is a critical point of E on $W_0^{1,p}(\Omega)$ then u_∞ is a solution to (2.3.2). Indeed, if u_∞ is a critical point then for arbitrary $v \in W_0^{1,p}(\Omega)$ it holds

$$\frac{d}{d\delta} E(u_\infty + \delta v)\Big|_{\delta=0} = a(\|\nabla u_\infty\|_p^p) \int_\Omega |\nabla u_\infty|^{p-2} \nabla u_\infty \nabla v - \langle f, v \rangle = 0.$$

Thus, u_∞ is a solution to (2.3.2) and a stationary point.

Theorem 2.3.2. *Let* (2.1.1) *holds, $f \in W^{-1,q}(\Omega)$, then $E(u)$ admits a global minimizer on $W_0^{1,p}(\Omega)$.*

Proof. To prove this theorem, we will use the direct method of calculus of variations. We claim that E is coercive and bounded from below. Indeed, Hölder's and Poincaré's inequalities imply $|\langle f, u \rangle| \le |f|_{-1,q} \|\nabla u\|_p$, therefore,

$$E(u) = \frac{1}{p} A(\|\nabla u\|_p^p) - \langle f, v \rangle \ge \frac{\lambda}{p} \|\nabla u\|_p^p - |f|_{-1,q} \|\nabla u\|_p. \tag{2.3.11}$$

Since $p > 1$ the coerciveness follows. Now coming back to (2.3.11) and using Young's inequality, we obtain

$$E(u) \ge \frac{\lambda}{p} \|\nabla u\|_p^p - \frac{(|f|_{-1,q})^q}{\lambda^{\frac{q}{p}} q} - \frac{\lambda}{p} \|\nabla u\|_p^p = -\frac{1}{q} \Big(\frac{|f|_{-1,q}}{\lambda^{\frac{1}{p}}} \Big)^q. \tag{2.3.12}$$

Thus, E is also bounded from below.

Let $u_n \in W_0^{1,p}(\Omega)$ be a minimizing sequence of E. From (2.3.11) it follows that u_n is bounded in $W_0^{1,p}(\Omega)$. Hence, for some $u_\infty \in W_0^{1,p}(\Omega)$, we have $u_n \rightharpoonup u_\infty$ in $W_0^{1,p}(\Omega)$. Next, we show that E is weakly lower semicontinuous on $W_0^{1,p}(\Omega)$. In fact, it holds that

$$\varliminf_{n\to\infty} \|\nabla u_n\|_p^p \ge \|\nabla u_\infty\|_p^p$$

(the norm is weakly lower semicontinuous). Considering a subsequence u_{n_k} such that

$$\varliminf_{n\to\infty} \|\nabla u_n\|_p^p = \lim_{k\to\infty} \|\nabla u_{n_k}\|_p^p$$

and due to the fact that u_{n_k} is a minimizing sequence, we see

$$\begin{aligned}\inf_{W_0^{1,p}(\Omega)} E(u) = \lim_k E(u_{n_k}) = \frac{1}{p} \int_0^{\lim \|\nabla u_{n_k}\|_p^p} & a(s) ds - \langle f, u_\infty \rangle \\ \ge \frac{1}{p} \int_0^{\|\nabla u_\infty\|_p^p} & a(s) ds - \langle f, u_\infty \rangle = E(u_\infty),\end{aligned}$$

which implies u_∞ is a minimizer of E on $W_0^{1,p}(\Omega)$. Therefore, the result follows. $\square$

Note that the minimizer might be not unique.

2.4 Remarks on the stationary points

Suppose first we are in case of Figure 2.4.2, then we have:

Theorem 2.4.1. *Let u_1 be the stationary point corresponding to μ_1 such that*

$$a(\mu) < \|\nabla\varphi\|_p^{p-1}\mu^{\frac{1}{p}-1} \quad \forall\mu \in (\underline{\mu}, \mu_1), \tag{2.4.1}$$

$$a(\mu) > \|\nabla\varphi\|_p^{p-1}\mu^{\frac{1}{p}-1} \quad \forall\mu \in (\mu_1, \overline{\mu}). \tag{2.4.2}$$

Then u_1 is a local minimizer for E. More precisely one has $E(u_1) < E(u)$ $\forall u \neq u_1$, $\|\nabla u\|_p^p \in (\underline{\mu}, \overline{\mu})$.

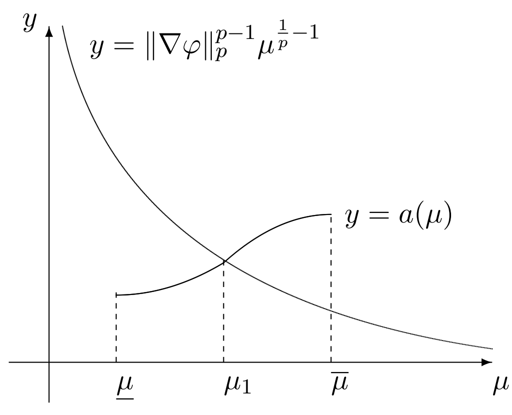


Figure 2.4.2

Proof. Recall that by Theorem 2.3.1, we have that

$$\mu_1 = \|\nabla u_1\|_p^p, \quad u_1 = \frac{\varphi}{a(\mu_1)^{\frac{1}{p-1}}}. \tag{2.4.3}$$

(i) Suppose $\|\nabla u\|_p^p > \mu_1$. Then from (2.1.7), (2.4.2), we have

$$\begin{aligned} E(u) - E(u_1) &= \frac{1}{p}\int_{\|\nabla u_1\|_p^p}^{\|\nabla u\|_p^p} a(s)ds - \langle f, u\rangle + \langle f, u_1\rangle \\ &> \frac{1}{p}\|\nabla\varphi\|_p^{p-1}\int_{\|\nabla u_1\|_p^p}^{\|\nabla u\|_p^p} s^{\frac{1}{p}-1}ds - \langle f, u\rangle + \langle f, u_1\rangle \\ &= \|\nabla\varphi\|_p^{p-1}\|\nabla u\|_p - \|\nabla\varphi\|_p^{p-1}\|\nabla u_1\|_p - \langle f, u\rangle + \langle f, u_1\rangle. \end{aligned} \tag{2.4.4}$$

From (2.3.3) and using Hölder's inequality, we see

$$
\begin{aligned}
|\langle f, u\rangle| &= \Big| \int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi\nabla u dx\Big| \\
&\le \Big(\int_\Omega |\nabla u|^p dx\Big)^{\frac{1}{p}} \Big(\int_\Omega |\nabla\varphi|^{q(p-1)}dx\Big)^{\frac{1}{q}} = \|\nabla u\|_p \|\nabla\varphi\|_p^{p-1},
\end{aligned} \tag{2.4.5}
$$

where $q = \frac{p}{p-1}$. Now, by (2.3.4) and (2.4.3), we obtain

$$
\begin{aligned}
\langle f, u_1\rangle &= \int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi\nabla u_1 dx \\
&= \int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi \frac{\nabla\varphi}{a(\mu_1)^{\frac{1}{p-1}}} dx = \|\nabla\varphi\|_p^p \frac{\|\nabla u_1\|_p}{\|\nabla\varphi\|_p} = \|\nabla\varphi\|_p^{p-1}\|\nabla u_1\|_p. \quad (2.4.6)
\end{aligned}
$$

Hence, combining (2.4.4) – (2.4.6), we derive $E(u) > E(u_1)$ for $\|\nabla u\|_p^p > \mu_1$.

(*ii*) Suppose now $\|\nabla u\|_p^p < \mu_1$. Then as above, we get

$$
E(u) - E(u_1) = -\frac{1}{p}\int_{\|\nabla u\|_p^p}^{\|\nabla u_1\|_p^p} a(s)ds - \langle f, u\rangle + \langle f, u_1\rangle,
$$

and by (2.4.1), (2.4.5), (2.4.6), we can conclude

$$
\begin{aligned}
E(u) - E(u_1) &> -\frac{1}{p}\|\nabla\varphi\|_p^{p-1}\int_{\|\nabla u\|_p^p}^{\|\nabla u_1\|_p^p} s^{\frac{1}{p}-1} ds - \langle f, u\rangle + \langle f, u_1\rangle \\
&= -\|\nabla\varphi\|_p^{p-1}\|\nabla u_1\|_p + \|\nabla\varphi\|_p^{p-1}\|\nabla u\|_p - \langle f, u\rangle + \langle f, u_1\rangle \ge 0. \quad (2.4.7)
\end{aligned}
$$

Thus, we have $E(u) > E(u_1)$ for $\|\nabla u\|_p^p \in (\underline{\mu}, \overline{\mu})$, $u \neq u_1$. □

Remark 2.4.1. If $u \neq u_1$ one does not have necessarily $\|\nabla u\|_p^p \neq \|\nabla u_1\|_p^p = \mu_1$ and the proof of the theorem is incomplete. But if $\|\nabla u\|_p^p = \|\nabla u_1\|_p^p$ one has (see above) $0 \le E(u) - E(u_1) = \langle f, u - u_1\rangle$. If this last quantity is vanishing, we will show in Lemma 5.2 that $u = u_1$.

Remark 2.4.2. If one assumes

$$
a(\mu) \le \|\nabla\varphi\|_p^{p-1}\mu^{\frac{1}{p}-1} \quad \forall\mu \le \mu_1,
$$

$$
a(\mu) \ge \|\nabla\varphi\|_p^{p-1}\mu^{\frac{1}{p}-1} \quad \forall\mu \ge \mu_1.
$$

Then, one gets only $E(u) \ge E(u_1)$. Thus, E can posses infinitely many global minimizers (see Figure 2.3.1d).

Lemma 2.4.2. *Let u_2 be the stationary point corresponding to μ_2 such that*

$$
a(\mu) > \|\nabla\varphi\|_p^{p-1}\mu^{\frac{1}{p}-1} \quad \forall\mu \in (\underline{\mu}, \mu_2), \tag{2.4.8}
$$

$$a(\mu) < \|\nabla\varphi\|_p^{p-1}\mu^{\frac{1}{p}-1} \quad \forall\mu \in (\mu_2, \overline{\mu}) \tag{2.4.9}$$

(see Figure 2.4.3). Then u_2 is a point of local maximum for E in the direction of φ, where φ is the solution of the problem (2.3.3). More precisely one has $E(u_2) > E(u_2 + \delta\varphi)$, for every $\delta \neq 0$ such that

$$\delta \geq -\frac{1}{a(\mu_2)^{\frac{1}{p-1}}} \quad , \quad \|\nabla(u_2 + \delta\varphi)\|_p^p \in (\underline{\mu}, \overline{\mu}).$$

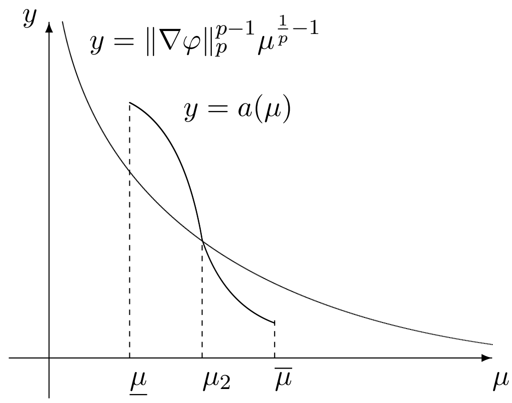


Figure 2.4.3

Proof. As above by Theorem 2.3.1, we have that

$$\mu_2 = \|\nabla u_2\|_p^p, \quad u_2 = \frac{\varphi}{a(\mu_2)^{\frac{1}{p-1}}}. \tag{2.4.10}$$

(i) Let us first assume that $\|\nabla(u_2 + \delta\varphi)\|_p^p > \mu_2$. Then from (2.1.7), (2.3.3), (2.4.9), we have

$$\begin{aligned} E(u_2 + \delta\varphi) - E(u_2) &= \frac{1}{p}\int_{\|\nabla u_2\|_p^p}^{\|\nabla(u_2+\delta\varphi)\|_p^p} a(s)ds - \delta\langle f, \varphi\rangle \\ &< \frac{1}{p}\|\nabla\varphi\|_p^{p-1}\int_{\|\nabla u_2\|_p^p}^{\|\nabla(u_2+\delta\varphi)\|_p^p} s^{\frac{1}{p}-1}ds - \delta\|\nabla\varphi\|_p^p \\ &= \|\nabla\varphi\|_p^{p-1}\big(\|\nabla(u_2 + \delta\varphi)\|_p - \|\nabla u_2\|_p\big) - \delta\|\nabla\varphi\|_p^p \\ &= \|\nabla\varphi\|_p^{p-1}\Big(\Big|\frac{1}{a(\mu_2)^{\frac{1}{p-1}}} + \delta\Big|\|\nabla\varphi\|_p - \frac{\|\nabla\varphi\|_p}{a(\mu_2)^{\frac{1}{p-1}}}\Big) - \delta\|\nabla\varphi\|_p^p = 0, \end{aligned} \tag{2.4.11}$$

if $\frac{1}{a(\mu_2)^{\frac{1}{p-1}}} + \delta \geq 0$. Thus, it holds that

$$E(u_2 + \delta\varphi) < E(u_2) \quad \text{for} \quad \|\nabla(u_2 + \delta\varphi)\|_p^p > \mu_2.$$

(ii) Suppose now $\|\nabla(u_2 + \delta\varphi)\|_p^p < \mu_2$. Then similarly, from (2.1.7), (2.3.3), (2.4.8), we get

$$\begin{aligned} E(u_2 + \delta\varphi) - E(u_2) &= -\frac{1}{p}\int_{\|\nabla(u_2+\delta\varphi)\|_p^p}^{\|\nabla u_2\|_p^p} a(s)ds - \delta\langle f, \varphi\rangle \\ &< -\|\nabla\varphi\|_p^{p-1}\big(\|\nabla u_2\|_p - \|\nabla(u_2 + \delta\varphi)\|_p\big) - \delta\|\nabla\varphi\|_p^p = 0 \end{aligned} \tag{2.4.12}$$

as in part (i). Hence,

$$E(u_2 + \delta\varphi) < E(u_2) \quad \text{for} \quad \|\nabla(u_2 + \delta\varphi)\|_p^p < \mu_2. \qquad \square$$

Lemma 2.4.3. *Let u be a solution to the problem* (2.3.2)*. Suppose that* (2.0.2) *holds and that $\psi \in W_0^{1,p}(\Omega)$, $\psi \neq 0$ is such that*

$$\langle f, \psi\rangle = 0. \tag{2.4.13}$$

Then

$$E(u + \psi) > E(u), \tag{2.4.14}$$

i.e., u is a point of minimum for E in any direction of the hyperplane defined by (2.4.13)*.*

Proof. Let us consider ψ which satisfies (2.4.13). Then for $\|\nabla(u + \psi)\|_p > \|\nabla u\|_p$ from (2.1.1), we have

$$E(u + \psi) - E(u) = \frac{1}{p}\int_{\|\nabla u\|_p^p}^{\|\nabla(u+\psi)\|_p^p} a(s)ds > 0.$$

Hence, it remains to prove that $\|\nabla(u + \psi)\|_p > \|\nabla u\|_p$. Due to (2.4.13) and since $a > 0$, we get

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\psi dx = 0.$$

Then, we see

$$\begin{aligned} \|\nabla(u + \psi)\|_p^p - \|\nabla u\|_p^p &= \int_0^1 \frac{d}{ds}\int_\Omega |\nabla(u + s\psi)|^p dxds \\ &= p\int_0^1\int_\Omega |\nabla(u + s\psi)|^{p-2}\nabla(u + s\psi)\nabla\psi dxds \\ &= p\int_0^1\int_\Omega \big(|\nabla(u + s\psi)|^{p-2}\nabla(u + s\psi) - |\nabla u|^{p-2}\nabla u\big)\nabla\psi dxds. \end{aligned}$$

From Lemma A.1, we have

$$\big(|\nabla(u+s\psi)|^{p-2}\nabla(u+s\psi)-|\nabla u|^{p-2}\nabla u\big)\nabla(s\psi) \geq c_p\big(|\nabla(u+s\psi)|+|\nabla u|\big)^{p-2}|\nabla(s\psi)|^2.$$

This shows that $\|\nabla(u+\psi)\|_p-\|\nabla u\|_p\geq 0$. If the equality holds then

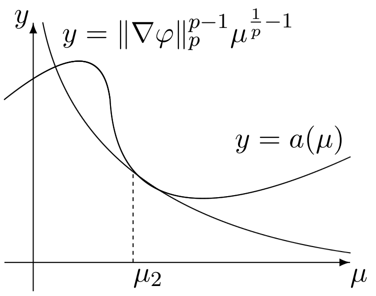
$$\big(|\nabla(u+s\psi)|+|\nabla u|\big)^{p-2}|\nabla\psi|^2=0 \ \text{ a.e. } x\in\Omega,\ s\in(0,1).$$

This implies that for $|\nabla u|=0$, we have $|\nabla\psi|=0$ and for $|\nabla u|\neq 0$ as well. Thus $\psi=0$, which contradicts our assumptions. This completes the proof of the theorem. □

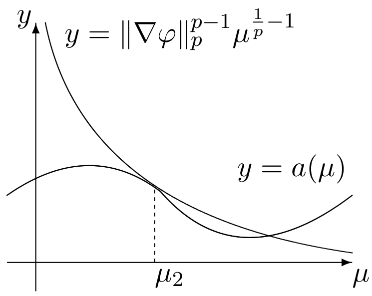
Theorem 2.4.4. *Let $f\not\equiv 0$, (2.0.2) holds, u_2 be a solution to (2.3.2) such that (2.4.8), (2.4.9) hold (see Figure 2.4.3, u_2 corresponds to μ_2). Then u_2 is a saddle point for the energy (2.1.7).*

Proof. The statement of the theorem is a consequence of Lemmas 2.4.2 and 2.4.3. □

Remark 2.4.3. The same situation occurs if the graph of a is not crossing the graph of y and touching it (see Figure 2.4.4).



(a)



(b)

Figure 2.4.4

Theorem 2.4.5. *Let u^* be a solution of the problem (2.3.2) corresponding to the solutions μ^* of the equation (2.3.4). Let*

$$y(s)=\|\nabla\varphi\|_p^{p-1}s^{\frac{1}{p}-1}, \tag{2.4.15}$$

then one has

$$E(u^*)=\frac{1}{p}\int_0^{\mu^*}(a(s)-y(s))ds.$$

Proof. From (2.1.7) one has

$$E(u^*) = \frac{1}{p}\int_0^{\|\nabla u^*\|_p^p} a(s)ds - \langle f, u^*\rangle.$$

Due to the definition of u^* (see (2.4.3)), we get

$$\frac{1}{p}\int_0^{\mu^*} y(s)ds = \frac{1}{p}\|\nabla\varphi\|_p^{p-1}\int_0^{\|\nabla u^*\|_p^p} s^{\frac{1}{p}-1}ds = \|\nabla\varphi\|_p^{p-1}\|\nabla u^*\|_p = \langle f, u^*\rangle.$$

(see (2.4.6)). Hence, the result follows. □

Corollary 2.4.6. *Let u_1, u_2 be two solutions of the problem* (2.3.2) *corresponding to the solutions $\mu_1 < \mu_2$ of the equation* (2.3.4) *respectively. Then one has*

$$E(u_1) - E(u_2) = -\frac{1}{p}\int_{\mu_1}^{\mu_2}(a(s) - y(s))ds =: -\frac{1}{p}A_{12} \tag{2.4.16}$$

and

$$A_{12} > 0 \;\Rightarrow\; E(u_1) < E(u_2);$$

$$A_{12} < 0 \;\Rightarrow\; E(u_2) < E(u_1);$$

$$A_{12} = 0 \;\Rightarrow\; E(u_1) = E(u_2).$$

Corollary 2.4.7. *Let u_1 and u_2 be two solutions of the problem* (2.3.2) *corresponding to the solutions $\mu_1 < \mu_2$ of the equation* (2.3.4)*. If we assume that*

$$a(\mu) > y(\mu) \quad \textit{for} \quad \mu_1 < \mu < \mu_2 \tag{2.4.17}$$

$$(\textit{resp.} \quad a(\mu) < y(\mu), \quad a(\mu) = y(\mu))\,, \tag{2.4.18}$$

then

$$E(u_1) < E(u_2) \quad (\textit{resp.}\ E(u_1) > E(u_2),\ E(u_1) = E(u_2)).$$

Corollary 2.4.8. *The absolute minimum of E corresponds to a point μ_∞ such that*

$$\int_{\mu_\infty}^{\mu}(a(s) - y(s))ds \geq 0, \quad \forall\mu > \mu_\infty,\ \mu \textit{ corresponding to a stationary point,}$$

$$\int_{\mu}^{\mu_\infty}(a(s) - y(s))ds \leq 0, \quad \forall\mu < \mu_\infty,\ \mu \textit{ corresponding to a stationary point.}$$

Therefore, due to Theorem 2.4.5 and its corollaries, we can compare the energy at any two different stationary points and we can find a global minimizer of the energy $E(u)$.

Example 2.4.1. Let u_i, $i = 1, 2, 3$ be solutions of the problem (2.3.2) corresponding to the solutions μ_i, $i = 1, 2, 3$ of the equation (2.3.4) such as on Figure 2.4.5. Then by Corollary 2.4.7, we get that $E(u_1) < E(u_2)$, $E(u_3) < E(u_2)$. It is left to compare the energy at the points u_1 and u_3. By Corollary 2.4.6, we see that

$$E(u_1) - E(u_3) = -\tfrac{1}{p}A_{13} = -\tfrac{1}{p}\left(|A_{12}| - |A_{23}|\right) < 0,$$

where

$$A_{ij} := \int_{\mu_i}^{\mu_j} (a(s) - y(s))ds, \quad i = 1, 2,\ j = 2, 3. \tag{2.4.19}$$

Hence, u_1 is a global minimizer of the energy $E(u)$ defined by (2.1.7).

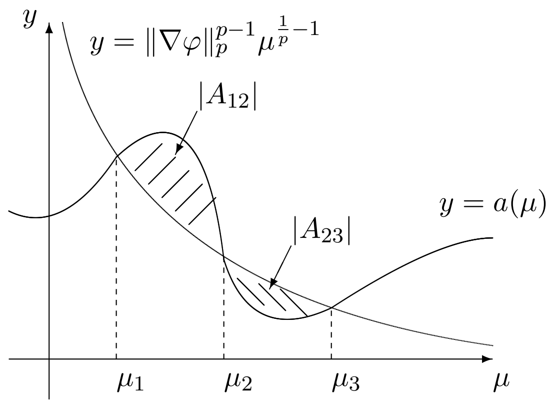


Figure 2.4.5: Several solutions

Remark 2.4.4. We label the solutions to (2.3.4) as $\mu_1 < \mu_2 < \ldots < \mu_N$ with the convention that we choose only one point μ_i in the interval $(\underline{\mu}_i, \overline{\mu}_i)$, when the solutions consist of one interval $(\underline{\mu}_i, \overline{\mu}_i)$ (see Figure 2.4.6). We denote by $\{u\}_1, \{u\}_2, \ldots \{u\}_N$ the sets of solutions of (2.3.2), corresponding to $\mu_1 < \mu_2 < \ldots < \mu_N$ solutions of (2.3.4). Then due to our convention, we see that $\{u\}_i$ can consist of one point or infinitely many points.

By Corollary 2.4.7 for arbitrary $u \in \{u\}_i$, $i \in I := \{1, \ldots, N\}$ it holds that $E(u) = E_i$, $i \in I$. Therefore, in the case when the stationary problem (2.3.2) is having infinitely many solutions, the energy (2.1.7) can have a unique, several or infinitely many global minimizers.

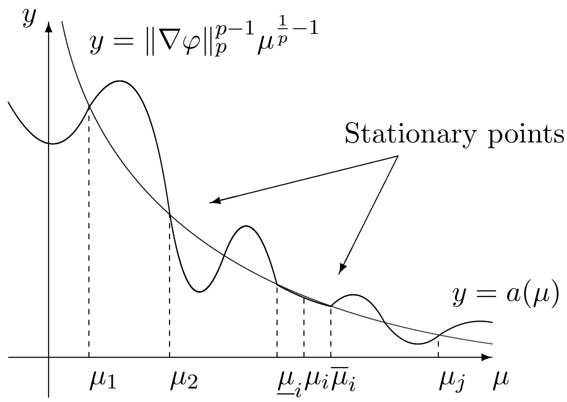


Figure 2.4.6: Infinitely many solutions

2.5 Asymptotic behaviour

We start with a lemma:

Lemma 2.5.1. *Let u be a weak solution to* (2.0.1) *and suppose that* (2.1.1) *holds. There exists a sequence t_k such that $u_k = u(\cdot, t_k) \to u_\infty$ in $W_0^{1,p}(\Omega)$ as $t_k \to +\infty$, where u_∞ is a stationary point.*

Proof. Taking $v = u_t$ in (2.1.2), we obtain

$$a(||\nabla u||_p^p)\int_\Omega |\nabla u|^{p-2}\nabla u\nabla u_t dx - \langle f, u_t\rangle = -|\partial_t u|_2^2,$$

$$\partial_t E(u) = -|\partial_t u|_2^2 \leq 0.$$

Hence, $E(u(t)) \leq E(u_0)$ and $E(u(t))$ decreases with the time. Remark that from (2.3.11), we have that $||\nabla u||_p^p$ is uniformly bounded in t. Since E is also bounded from below (see (2.3.12)), then it follows

$$E(u(t)) \to E_\infty \tag{2.5.1}$$

(E_∞ is some constant). From above, we get

$$E(u(t)) - E(u(s)) = -\int_s^t |\partial_t u|_2^2(\xi)d\xi, \qquad \int_s^\infty |\partial_t u|_2^2(\xi)d\xi < +\infty,$$

which implies for a sequence t_k that $\partial_t u(\cdot, t_k) \to 0$ in $L^2(\Omega)$. From the equation in (2.1.2) with $u_k = u(\cdot, t_k)$, we obtain

$$\int_\Omega \partial_t u(\cdot, t_k)u_k dx + a(||\nabla u_k||_p^p)\int_\Omega |\nabla u_k|^p dx = \langle f, u_k\rangle. \tag{2.5.2}$$

We can show as in (2.1.5) that $|u(t)|_2^2 \leq C$. Up to a subsequence it holds that $u(\cdot,t_k) = u_k \rightharpoonup u_\infty$ in $W_0^{1,p}(\Omega)$, $u_k \rightharpoonup u_\infty$ in $L^2(\Omega)$, $\|\nabla u_k\|_p^p \to l_\infty$, $g_k = |\nabla u_k|^{p-2}\nabla u_k \rightharpoonup g_\infty$ in $(L^q(\Omega))^n$. Passing to the limit in (2.5.2), we see

$$a(l_\infty)l_\infty = \langle f, u_\infty\rangle. \tag{2.5.3}$$

Next taking $v \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$, we get

$$\int_\Omega \partial_t u(\cdot,t_k) v dx + a(||\nabla u_k||_p^p) \int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla v dx = \langle f, v\rangle.$$

Passing to the limit, we derive

$$a(l_\infty)\int_\Omega g_\infty \nabla v dx = \langle f, v\rangle. \tag{2.5.4}$$

Since $a > 0$ combining (2.5.3), (2.5.4), we can conclude that

$$l_\infty = \int_\Omega g_\infty \nabla u_\infty dx.$$

We claim that $u_k \to u_\infty$ strongly in $W_0^{1,p}(\Omega)$. Indeed, for $p \geq 2$ there exists a constant $C_p > 0$ such that

$$\begin{aligned}\chi_k = \int_\Omega \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_\infty|^{p-2}\nabla u_\infty\right)\nabla(u_k - u_\infty)dx \\ \geq C_p\int_\Omega |\nabla(u_k - u_\infty)|^p dx.\end{aligned}$$

Developing

$$\begin{aligned}\chi_k = \int_\Omega |\nabla u_k|^p dx - \int_\Omega g_k \nabla u_\infty dx - \int_\Omega |\nabla u_\infty|^{p-2}\nabla u_\infty \nabla u_k dx + \int_\Omega |\nabla u_\infty|^p dx \\ \to l_\infty - \int_\Omega g_\infty \nabla u_\infty dx - \int_\Omega |\nabla u_\infty|^p dx + \int_\Omega |\nabla u_\infty|^p dx = 0.\end{aligned}$$

This implies

$$l_\infty = \lim_k ||\nabla u_k||_p^p = ||\nabla u_\infty||_p^p,\ g_\infty = |\nabla u_\infty|^{p-2}\nabla u_\infty.$$

Hence, u_∞ is a stationary point.

To show that $u_k \to u_\infty$ in $W_0^{1,p}(\Omega)$ strongly in case $1 < p < 2$, it is enough to notice that by Lemma A.1 one has

$$c_p\int_\Omega |\nabla(u_k - u_\infty)|^2\big(|\nabla u_k| + |\nabla u_\infty|\big)^{p-2}dx \leq \chi_k \to 0.$$

Writing

$$\int_\Omega |\nabla(u_k - u_\infty)|^p dx = \int_\Omega |\nabla(u_k - u_\infty)|^p \big(|\nabla u_k| + |\nabla u_\infty|\big)^{\frac{(p-2)p}{2}} \big(|\nabla u_k| + |\nabla u_\infty|\big)^{\frac{(2-p)p}{2}} dx$$

and using Hölder's inequality with $\frac{2}{p}$, $\frac{2}{2-p}$ it comes

$$\int_\Omega |\nabla(u_k - u_\infty)|^p dx \le \Big(\int_\Omega |\nabla(u_k - u_\infty)|^2 \big(|\nabla u_k| + |\nabla u_\infty|\big)^{p-2} dx\Big)^{\frac{p}{2}} \times \Big(\int_\Omega \big(|\nabla u_k| + |\nabla u_\infty|\big)^p dx\Big)^{\frac{2-p}{2}} \le C\chi_k \to 0.$$

This completes the proof of the lemma. □

Corollary 2.5.2. *Suppose that E admits a unique global minimizer u_∞ (u_∞ is also a solution to the problem* (2.3.2)) *and that the initial value u_0 of* (2.1.2) *satisfies $E(u_0) < E(u_i)$ for any stationary point $u_i \neq u_\infty$. Then $u(\cdot, t) \to u_\infty$ in $W_0^{1,p}(\Omega)$ as $t \to +\infty$.*

Proof. Recall that we have $E(u) \le E(u_0) < E(u_i)$, $u_i \neq u_\infty$. Then by Lemma 2.5.1 and (2.5.1), we get $E(u(t)) \to E(u_\infty)$, where u_∞ is the global minimizer of E and a solution of the problem (2.3.2). Due to the fact that $u(t)$ is uniformly bounded in $W_0^{1,p}(\Omega)$ for some subsequence, we have $u(\cdot, t_k) \rightharpoonup v_\infty$ in $W_0^{1,p}(\Omega)$. Then by the weak lower semicontinuity of E (see the proof of Theorem 2.3.2), we obtain

$$E(u_\infty) = \lim_{t_k \to \infty} E(u(t_k)) \ge E(v_\infty).$$

Since u_∞ is a unique global minimizer of E, then it holds that $E(u_\infty) < E(v_\infty)$ for $u_\infty \neq v_\infty$, hence $u_\infty = v_\infty$. This holds for every subsequence and the convergence is in fact strong (see Lemma 2.5.1), therefore, the result follows. □

Remark 2.5.1. In the case where $a(s^p)s^{p-1}$ is increasing (see (2.2.5) and Remark 4.1) the problem has a single stationary point and for any initial data $u(\cdot, t)$ converges to this stationary point.

Theorem 2.5.3. *Let the assumptions of Theorem 2.1.1 hold and in addition a' is continuous, $2 \le p < +\infty$ and $f \in L^2(\Omega)$. Then for any $T > 0$ there exists a unique strong solution u to* (2.0.1) *such that*

$$u \in C([0,T]; W_0^{1,p}(\Omega)), \quad u_t,\ \nabla \cdot |\nabla u|^{p-2}\nabla u \in L^2(0,T; L^2(\Omega)). \tag{2.5.5}$$

Proof. Let $\lambda_1, \ldots, \lambda_n, \ldots$ be a basis in $W^{2,2(p-1)}(\Omega)$ consisting of eigenvalues of the problem

$$\lambda_i \in H_0^s(\Omega),\ (\lambda_i, v)_{H_0^s(\Omega)} = \mu_i(\lambda_i, v)_{L^2(\Omega)}\ \forall v \in H_0^s(\Omega), \tag{2.5.6}$$

where we choose s in such a way that $H_0^s(\Omega) \subset W^{2,2(p-1)}(\Omega)$. We will suppose that λ_i are orthonormal in $L^2(\Omega)$ ($W^{2,2(p-1)}(\Omega) \subset L^2(\Omega)$, since $p \geq 2$). Let now $u_n(t) = \sum_{i=1}^{n} \gamma_i(t)\lambda_i$ be a solution to (2.1.3) with $u_0 = \sum_i \beta_i \lambda_i$. Then γ_j satisfy the Cauchy problem (2.1.4) (see Theorem 2.1.1). By the existence theorem for the ordinary differential equations this Cauchy problem possesses a solution $\gamma_j \in C^2([0,\delta))$, $\delta > 0$. Recall that (see (2.1.11))

$$u_n' \in L^2(0,T; L^2(\Omega)) = L^2(Q_T), \quad Q_T = (0,T) \times \Omega.$$

Hence, we can differentiate (2.1.3) with respect to t and since

$$\begin{aligned}\frac{d}{dt}|\nabla u_n|^p = \frac{d}{dt}\big(|\nabla u_n|^2\big)^{\frac{p}{2}} = \frac{p}{2}\big(|\nabla u_n|^2\big)^{\frac{p}{2}-1}\frac{d}{dt}|\nabla u_n|^2 \\ = \frac{p}{2}|\nabla u_n|^{p-2} 2\nabla u_n \nabla u_n' = p|\nabla u_n|^{p-2}\nabla u_n \nabla u_n'\end{aligned}$$

we get

$$\begin{aligned}\int_\Omega u_n'' v dx + pa'(\|\nabla u_n\|_p^p)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u_n' dx \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v dx \\ + a(\|\nabla u_n\|_p^p)\int_\Omega (p-2)|\nabla u_n|^{p-4}\nabla u_n \nabla u_n' \nabla u_n \nabla v + |\nabla u_n|^{p-2}\nabla u_n' \nabla v dx = 0.\end{aligned}$$

Taking $v = u_n'$ and noting that the last term is nonnegative we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_n'|^2 dx \leq -pa'(\|\nabla u_n\|_p^p)\left(\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u_n' dx\right)^2. \tag{2.5.7}$$

From the first equation in (2.1.3) written with $v = u_n'$ we have

$$a(\|\nabla u_n\|_p^p)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u_n' dx = \int_\Omega f u_n' dx - \int_\Omega |u_n'|^2 dx$$

and from (2.5.7) follows

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_n'|^2 dx \leq p\frac{|a'(\|\nabla u_n\|_p^p)|}{a^2(\|\nabla u_n\|_p^p)}\left(\int_\Omega f u_n' dx - \int_\Omega |u_n'|^2 dx\right)^2.$$

Since $E(u_n)$ is uniformly bounded so is $\|\nabla u_n\|_p^p$. Due to the fact that $a \in C^1$ from Hölder's inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_n'|^2 dx \leq C\left(\int_\Omega |f|^2 dx + \int_\Omega |u_n'|^2 dx\right)\int_\Omega |u_n'|^2 dx. \tag{2.5.8}$$

Denote by $y_n(t) = |u_n'(t)|_2^2$. Integrating (2.5.8) we get

$$y_n(t) - y_n(s) \leq 2C\int_s^t \big(|f|_2^2 + y_n(\xi)\big) y_n(\xi) d\xi.$$

Passing to the limit in (2.1.12) as $T \to +\infty$ we obtain that

$$\int_0^{+\infty} y_n(s)ds < +\infty.$$

Hence, since $g(x) = 2C(|f|_2^2 x + x^2) > 0$ on $x > 0$ from Lemma A.5 we derive

$$y_n(t) \to 0 \text{ as } t \to +\infty.$$

Thus y_n remains bounded in time. Remark that

$$\nabla \cdot |\nabla u|^{p-2}\nabla u = |\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4} \sum_{i,j=1}^{n} u_{x_i}u_{x_j}u_{x_ix_j}.$$

Applying twice the Cauchy-Schwarz inequality we get

$$\int_\Omega |\nabla \cdot |\nabla u|^{p-2}\nabla u|^2 dx \le \frac{1}{2}\Big(\int_\Omega |\nabla u|^{2p-4}|\Delta u|^2 dx + (p-2)^2 \int_\Omega |\nabla u|^{2p-4} \sum_{i,j=1}^{n} u^2_{x_ix_j} dx\Big). \quad (2.5.9)$$

From Hölder's inequality with the exponents $\frac{p-1}{p-2}$, $p-1$ we get that

$$\int_\Omega |\nabla u|^{2p-4}|\Delta u|^2 dx \le \left(\int_\Omega |\nabla u|^{2(p-1)} dx\right)^{\frac{p-2}{p-1}} \left(\int_\Omega |\Delta u|^{2(p-1)} dx\right)^{\frac{1}{p-1}}.$$

We can estimate the second term in (2.5.9) in a similar way. Hence, since $\lambda_j \in W^{2,2(p-1)}(\Omega)$, we can multiply the first equation in (2.1.4) by $\lambda_j \nabla \cdot |\nabla u_n|^{p-2}\nabla u_n$, then integrating over Ω and summing in j we get

$$\int_\Omega \nabla \cdot |\nabla u_n|^{p-2}\nabla u_n u_n' dx = a(\|\nabla u_n\|_p^p) \sum_{j=1}^{n} \left(\int_\Omega \nabla \cdot |\nabla u_n|^{p-2}\nabla u_n \lambda_j dx\right)^2 + \sum_{j=1}^{n} \int_\Omega f\lambda_j dx \int_\Omega \nabla \cdot |\nabla u_n|^{p-2}\nabla u_n \lambda_j dx.$$

Since $\lambda_1, \dots, \lambda_n$ are orthonormal in $L^2(\Omega)$ the equality above can be written as

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u_n' dx + a(\|\nabla u_n\|_p^p)|P_n(\nabla \cdot |\nabla u_n|^{p-2}\nabla u_n)|_2^2 = -\int_\Omega P_n f \nabla \cdot |\nabla u_n|^{p-2}\nabla u_n dx,$$

where P_n denotes a projection operator from $L^2(\Omega)$ onto $[\lambda_1, \ldots, \lambda_n]$. Then from (2.1.1), Hölder's and Young's inequalities we get

$$\frac{1}{p}\frac{d}{dt}\|\nabla u_n\|_p^p + \lambda|P_n(\nabla\cdot|\nabla u_n|^{p-2}\nabla u_n)|_2^2 \le |(f, P_n(\nabla\cdot|\nabla u_n|^{p-2}\nabla u_n))|$$
$$\le |f|_2|P_n(\nabla\cdot|\nabla u_n|^{p-2}\nabla u_n)|_2 \le \frac{|f|_2^2}{2\lambda} + \frac{\lambda|P_n(\nabla\cdot|\nabla u_n|^{p-2}\nabla u_n)|_2^2}{2}.$$

Therefore, we obtain

$$\frac{1}{p}\frac{d}{dt}\|\nabla u_n\|_p^p + \frac{\lambda}{2}|P_n(\nabla\cdot|\nabla u_n|^{p-2}\nabla u_n)|_2^2 \le \frac{|f|_2^2}{2\lambda}.$$

And after integration in time

$$\frac{1}{p}\|\nabla u_n\|_p^p + \frac{\lambda}{2}\int_0^t |P_n(\nabla\cdot|\nabla u_n|^{p-2}\nabla u_n)|_2^2 dt \le \frac{1}{p}\|\nabla u_0\|_p^p + \frac{|f|_2^2 T}{2\lambda}. \tag{2.5.10}$$

From (2.1.11), (2.5.10) follow that we can find a subsequence of n such that

$$u_n' \rightharpoonup u' \text{ in } L^2(Q_T),$$
$$P_n(\nabla\cdot|\nabla u_n|^{p-2}\nabla u_n) \rightharpoonup \chi \text{ in } L^2(Q_T).$$

Notice that from the proof of Theorem 2.1.1 we know already that

$$\nabla\cdot|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup \nabla\cdot|\nabla u|^{p-2}\nabla u \text{ in } L^q(0,T;W^{-1,q}(\Omega)) \subset L^q(0,T;H^{-s}(\Omega)),$$
$$\|\nabla u_n\|_p^p \to \|\nabla u\|_p^p \text{ a.e. } t.$$

Let $w \in L^2(\Omega)$, then $P_n w \in [\lambda_1, \ldots, \lambda_n]$. Taking now in (2.1.3) $v = P_n w$ and passing to the limit ($P_n w \to w$ in $L^2(\Omega)$, see [45]), we obtain

$$\int_\Omega u' w dx - a(\|\nabla u\|_p^p)\int_\Omega \chi w dx = \int_\Omega f w dx \quad \forall w \in L^2(\Omega) \text{ in } \mathcal{D}'(0,T).$$

Remark that for $w \in H_0^s(\Omega)$ it holds that $P_n w \to w$ in $H_0^s(\Omega)$. Indeed,

$$w = \sum_{j=1}^\infty (\lambda_j, w)\lambda_j$$

and due to (2.5.6) we get that

$$\|w\|_{H_0^s(\Omega)}^2 = \sum_{j=1}^\infty |(\lambda_j, w)|^2 \mu_j < +\infty.$$

Then

$$\|P_n w - w\|_{H_0^s(\Omega)}^2 = \left\|\sum_{j=n+1}^\infty (\lambda_j, w)\lambda_j\right\|_{H_0^s(\Omega)}^2 = \sum_{j=n+1}^\infty |(\lambda_j, w)|^2 \mu_j \to 0.$$

Therefore for $w \in H_0^s(\Omega)$, $\varphi \in \mathcal{D}(0,T)$ we obtain

$$\int_0^T \int_\Omega \chi w\varphi dxdt = \lim_{n\to+\infty} \int_0^T \int_\Omega P_n(\nabla \cdot |\nabla u_n|^{p-2}\nabla u_n) w\varphi dxdt$$
$$= \lim_{n\to+\infty} \int_0^T \int_\Omega \nabla \cdot |\nabla u_n|^{p-2}\nabla u_n P_n w\varphi dxdt = \int_0^T \int_\Omega \nabla \cdot |\nabla u|^{p-2}\nabla u w\varphi dxdt.$$

Hence, $\chi = \nabla \cdot |\nabla u|^{p-2}\nabla u$ and u is a solution to (2.0.1) with

$$u_t - a(\|\nabla u\|_p^p)\nabla \cdot |\nabla u|^{p-2}\nabla u = f \text{ in } L^2(Q_T).$$

It remains to show that $u \in C([0,T]; W_0^{1,p}(\Omega))$. By rescaling the time as in (2.1.13) we reduce solving the problem (2.0.1) to solving the problem (2.1.14). Then (we keep denoting the solution by u) multiplying the first equation in (2.1.14) by u_t and integrating over Ω we get

$$\int_\Omega |u_t|^2 dx + \int_\Omega |\nabla u|^{p-2}\nabla u \nabla u_t dx = \int_\Omega \frac{f u_t}{a(\|\nabla u(\cdot,t)\|_p^p)} dx.$$

Using (2.1.1) and Hölder's and Young's inequalities we obtain

$$|u_t|_2^2 + \frac{1}{p}\frac{d}{dt}\|\nabla u\|_p^p \le \frac{1}{\lambda}|f|_2|u_t|_2 \le \frac{|f|_2^2}{2\lambda^2} + \frac{|u_t|_2^2}{2}.$$

Therefore, it holds that

$$\frac{d}{dt}\|\nabla u\|_p^p \le C|f|_2^2.$$

Integrating from t_0 to t we deduce

$$\|\nabla u(t)\|_p^p \le \|\nabla u(t_0)\|_p^p + C|f|_2^2(t - t_0).$$

Hence, letting $t \to t_0$ we get

$$\limsup_{t\to t_0} \|\nabla u(t)\|_p \le \|\nabla u(t_0)\|_p. \tag{2.5.11}$$

Recall that

$$\|\nabla u(t)\|_p \le C \quad \forall t \ge 0,$$

thus for a subsequence

$$\nabla u(t_k) \rightharpoonup \tilde{u} \text{ in } (L^p(\Omega))^n \text{ as } t_k \to t_0.$$

Note, that since $u \in C([0,T]; L^2(\Omega))$ (see Theorem 2.1.1) we have $u(t) \to u(t_0)$ in $L^2(\Omega)$. Then for $\varphi \in (\mathcal{D}(\Omega))^n$ we see

$$\int_\Omega \nabla u(t_k)\varphi dx = -\int_\Omega u(t_k)\nabla \cdot \varphi dx \to -\int_\Omega u(t_0)\nabla \cdot \varphi dx = \int_\Omega \nabla u(t_0)\varphi dx.$$

Thus we get that $\tilde{u} = \nabla u(t_0)$. Then by the weak lower semicontinuity of the norm we know that

$$\|\nabla u(t_0)\|_p \leq \liminf_{t_k \to t_0} \|\nabla u(t_k)\|_p.$$

Therefore, by (2.5.11) we see

$$\|\nabla u(t_k)\|_p^p \to \|\nabla u(t_0)\|_p^p \text{ as } t_k \to t_0,\ t_0 \geq 0. \tag{2.5.12}$$

Combining (2.5.12) and the fact that $\nabla u(t_k) \rightharpoonup \nabla u(t_0)$ in $(L^p(\Omega))^n$ we get that

$$\|\nabla(u(t_k) - u(t_0))\|_p^p \to 0 \text{ as } t_k \to t_0,\ t_0 \geq 0.$$

Since the limit is unique and this holds for every subsequence, hence, we get the result. Uniqueness follows by the uniqueness result for a weak solution. $\square$

Theorem 2.5.4. *Let $p \geq 2$, u_* be an isolated solution to the problem* (2.3.1)*, corresponding to the solution μ_* of the equation* (2.3.4)*. Assume that the function a' is continuous and*

$$\frac{p}{p-1} a'(\mu_*)\mu_* + a(\mu_*) = \delta > 0. \tag{2.5.13}$$

Then there exists $\varepsilon > 0$ such that if the initial value $u_0 \in \mathcal{N}_\varepsilon(u_*)$, where

$$\mathcal{N}_\varepsilon(u_*) := \left\{ u \in W_0^{1,p}(\Omega) : \|\nabla(u - u_*)\|_p < \varepsilon,\ E(u) < E(u_*) + \frac{\delta \varepsilon^p}{16(18)^{\frac{p}{2}}} \right\} \tag{2.5.14}$$

then

$$u(t) \to u_* \text{ in } W_0^{1,p}(\Omega). \tag{2.5.15}$$

Proof. Set $\mathcal{E}(s) = E(u_* + s(u - u_*))$. Then one has

$$\begin{aligned} E(u) - E(u_*) = \mathcal{E}(1) - \mathcal{E}(0) = \int_0^1 \mathcal{E}'(s)ds = \mathcal{E}'(0) + \int_0^1 (1-s)\mathcal{E}''(s)ds \\ = \int_0^1 (1-s)\mathcal{E}''(s)ds, \end{aligned} \tag{2.5.16}$$

since $\mathcal{E}'(0) = 0$ due to the fact that u_* is a stationary point.

Denote by $w = u - u_*$. After a simple computation we see that

$$\begin{aligned} \mathcal{E}''(s) = pa'(\|\nabla(u_* + sw)\|_p^p) \left(\int_\Omega |\nabla(u_* + sw)|^{p-2} \nabla(u_* + sw)\nabla w dx \right)^2 \\ + a(\|\nabla(u_* + sw)\|_p^p) \Big(\int_\Omega (p-2)|\nabla(u_* + sw)|^{p-4}(\nabla(u_* + sw)\nabla w)^2 \\ + |\nabla(u_* + sw)|^{p-2}|\nabla w|^2 dx \Big). \end{aligned} \tag{2.5.17}$$

On $a'(\mu) \geq 0$ since $p \geq 2$ one has

$$\mathcal{E}''(s) \geq a(\|\nabla(u_* + sw)\|_p^p) \int_\Omega |\nabla(u_* + sw)|^{p-2}|\nabla w|^2 dx. \tag{2.5.18}$$

Remark that by the Hölder and the Cauchy-Schwarz inequalities we have that

$$\begin{aligned}
&\Big(\int_\Omega |\nabla(u_*+sw)|^{p-2}\nabla(u_*+sw)\nabla w dx\Big)^2 \\
&\quad\le \frac{p-2+1}{p-1}\int_\Omega |\nabla(u_*+sw)|^{p-4}(\nabla(u_*+sw)\nabla w)^2 dx \int_\Omega |\nabla u_*+sw|^p dx \\
&\qquad\le \frac{1}{p-1}\Big((p-2)\int_\Omega |\nabla(u_*+sw)|^{p-4}(\nabla(u_*+sw)\nabla w)^2 dx \\
&\qquad\qquad + \int_\Omega |\nabla(u_*+sw)|^{p-2}|\nabla w|^2 dx\Big)\int_\Omega |\nabla u_*+sw|^p dx.
\end{aligned}$$

Therefore, on $a'(\mu)<0$ we get that

$$\begin{aligned}
\mathcal{E}''(s) \ge &\left(\frac{p}{p-1}a'(\|\nabla(u_*+sw)\|_p^p)\|\nabla(u_*+sw)\|_p^p + a(\|\nabla(u_*+sw)\|_p^p)\right) \\
&\times\Big(\int_\Omega (p-2)|\nabla(u_*+sw)|^{p-4}(\nabla(u_*+sw)\nabla w)^2 \\
&\qquad + |\nabla(u_*+sw)|^{p-2}|\nabla w|^2 dx\Big). \qquad (2.5.19)
\end{aligned}$$

From Lemma A.3 and the Hölder inequality for $s\in(0,1)$ we see that

$$\begin{aligned}
\big|\|\nabla(u_*+sw)\|_p^p - \|\nabla u_*\|_p^p\big| &\le ps\int_\Omega \big(|\nabla(u_*+sw)|+|\nabla u_*|\big)^{p-1}|\nabla w| dx \\
&\le p\big||\nabla(u_*+sw)|+|\nabla u_*|\big|_p^{p-1}\|\nabla w\|_p.
\end{aligned}$$

Hence, by the continuity of a' and due to the assumption (2.5.13) from (2.5.18) and (2.5.19) we can deduce that there exists $\eta>0$ such that

$$\|\nabla w\|_p \le \eta \quad \Rightarrow \quad \mathcal{E}''(s) \ge \frac{\delta}{2}\int_\Omega |\nabla(u_*+sw)|^{p-2}|\nabla w|^2 dx, \qquad (2.5.20)$$

i.e. by (2.5.16) and Lemma A.4

$$E(u)-E(u_*) \ge \frac{\delta}{2}\int_0^1 (1-s)\int_\Omega |\nabla(u_*+sw)|^{p-2}|\nabla w|^2 dxds \ge \frac{\delta}{16(18)^{\frac{p}{2}}}\|\nabla w\|_p^p. \qquad (2.5.21)$$

We choose $\varepsilon<\eta$ such that u_* is the unique stationary point in

$$B_\varepsilon = \{u : \|\nabla(u-u_*)\|_p < \varepsilon\}$$

(we can do this since u_* is an isolated stationary point) and $u_0\in\mathcal{N}_\varepsilon(u_*)$. We introduce the set A defined by

$$A = \{t\in[0,+\infty) \mid u(t)\in\mathcal{N}_\varepsilon(u_*)\}.$$

Since $u \in C([0,T]; W_0^{1,p}(\Omega))$ it is clear that A contains a neighbourhood of 0 and is open. Denote by t_∞ the point such that $t_\infty = Sup\{t \mid [0,t) \subset A\}$. Let t_n be a sequence in A such that $t_n \to t_\infty, t_n < t_\infty$. Since $u \in C([0,T]; W_0^{1,p}(\Omega))$ one has

$$\|\nabla(u(t_\infty) - u_*)\|_p \leq \varepsilon < \eta.$$

Hence using the fact that E is decreasing along the trajectories and (2.5.20), (2.5.21) we deduce that

$$\frac{\delta}{16(18)^{\frac{p}{2}}} \|\nabla(u(t_\infty) - u_*)\|_p^p \leq E(u(t_\infty)) - E(u_*) < \frac{\delta}{16(18)^{\frac{p}{2}}} \varepsilon^p,$$

i.e. $t_\infty \in A$ and since A is open we get a contradiction with the definition of t_∞. Thus t_∞ is not finite and $A = [0, \infty)$. So $u(t) \in \mathcal{N}_\varepsilon(u_*)$ for all t. For a subsequence we have that

$$u(t_k) \to u_\infty \ \text{ in } W_0^{1,p}(\Omega),$$

where u_∞ is a stationary point. Since u_* is the only stationary point in B_ε then $u_\infty = u_*$ and the result follows. $\square$

Remark 2.5.2. The assumption (2.5.13) is equivalent to

$$a'(\mu_*) > \frac{(1-p)a(\mu_*)}{p\mu_*} = y'(\mu_*).$$

Therefore,

$$\lim_{\mu \to \mu_*} \frac{a(\mu) - a(\mu_*) + y(\mu_*) - y(\mu)}{\mu - \mu_*} > 0$$

and it holds that there exists $\alpha > 0$ such that

$$(a(\mu) - y(\mu))(\mu - \mu_*) > 0 \quad \forall \mu \in (\mu_* - \alpha, \mu_* + \alpha),\ \mu \neq \mu_*.$$

Thus from Remark 2.5.2 we see that the stationary point u_* corresponds to the isolated local minimizer of the energy E (see Theorem 2.4.1 and Figure 2.4.2). And Theorem 2.5.4 can be reformulated as follows:

Theorem 2.5.5. *The isolated local minimizers of the energy E defined by* (2.1.7) *are asymptotically stable.*

Chapter 3

Nonlocal p-Laplace equations depending on the integral term

In this chapter we consider the problem of finding $u = u(x,t)$ solution to

$$\begin{cases} u_t - \nabla \cdot a(l(u(t)))|\nabla u|^{p-2}\nabla u = f & \text{in } \ \Omega \times (0,T), \\ u = 0 & \text{on } \ \Gamma \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \ \Omega, \end{cases} \tag{3.0.1}$$

where $u(t) = u(\cdot,t)$ and $l(u(t))$ is defined by

$$l(u(t)) := \int_\Omega g(x)u(x,t)dx. \tag{3.0.2}$$

Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 1$ with Lipschitz boundary Γ, $1 < p < +\infty$. We will assume that

$$a \text{ is continuous, } a(\xi) > 0, \ \forall \xi \in \mathbb{R}, \tag{3.0.3}$$

and

$$g \in L^q(\Omega), \ u_0 \in L^2(\Omega), \ f = f(x) \in W^{-1,q}(\Omega), \ \frac{1}{p} + \frac{1}{q} = 1. \tag{3.0.4}$$

3.1 Existence and uniqueness

Theorem 3.1.1. *Let the assumptions above hold and there exist two constants λ, Λ such that*

$$0 < \lambda \leq a(\xi) \leq \Lambda, \quad \forall \xi \in \mathbb{R}. \tag{3.1.1}$$

Then, for any $T > 0$ there exists u solution to

$$\begin{cases} u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap C([0,T];L^r(\Omega)), r = \min\{2,p\},\ u_t \in L^q(0,T;W^{-1,q}(\Omega)), \\ u(\cdot,0) = u_0, \\ \langle u_t, v\rangle + \int_\Omega a(l(u(t)))|\nabla u|^{p-2}\nabla u \nabla v dx = \langle f, v\rangle \ \forall v \in W_0^{1,p}(\Omega) \ in \ \mathcal{D}'(0,T), \end{cases} \tag{3.1.2}$$

where $\langle \cdot,\cdot\rangle$ denotes the pairing between $W_0^{1,p}(\Omega)$ and its dual $W^{-1,q}(\Omega)$, $\mathcal{D}'(0,T)$ is the space of distributions on $(0,T)$.

Proof. The proof will be based on the Schauder fixed point theorem. For $w \in L^p(0,T;L^p(\Omega))$ the mapping

$$t \mapsto a(l(w(\cdot,t))$$

is measurable, because a and l are continuous and there exists a unique solution u to

$$\begin{cases} u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)),\ u_t \in L^q(0,T;W^{-1,q}(\Omega)), \\ \langle u_t, v\rangle + \int_\Omega a(l(w))|\nabla u|^{p-2}\nabla u \nabla v dx = \langle f, v\rangle \ \forall v \in W_0^{1,p}(\Omega) \ in \ \mathcal{D}'(0,T), \\ u(\cdot,0) = u_0. \end{cases} \tag{3.1.3}$$

Then we would like to show that the mapping

$$w \mapsto T(w) = u$$

has a fixed point. To do this, first, we show that T maps $L^p(0,T;L^p(\Omega))$ into itself. Taking $v = u$ in (3.1.3) we have

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + a(l(w))\int_\Omega |\nabla u|^p dx = \langle f, u\rangle.$$

Then from (3.1.1), Hölder's and Young's inequalities we get

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + \lambda\|\nabla u\|_p^p \le |f|_{-1,q}\|\nabla u\|_p \le C_\varepsilon |f|_{-1,q}^q + \varepsilon\|\nabla u\|_p^p.$$

Taking $\varepsilon = \dfrac{\lambda}{2}$ and integrating over $(0,t)$ we obtain

$$\frac{|u(t)|_2^2}{2} + \frac{\lambda}{2}\int_0^t \|\nabla u\|_p^p dt \le C_\varepsilon \int_0^t |f|_{-1,q}^q dt + \frac{|u(0)|_2^2}{2}.$$

This and Poincaré's inequality imply that

$$\int_0^t |u|_p^p dt \le C_p \int_0^t \|\nabla u\|_p^p dt \le C, \tag{3.1.4}$$

$$|u(t)|_2^2 \le C, \tag{3.1.5}$$

i.e.

$$u \in L^p(0,T;W_0^{1,p}(\Omega)), \quad u \in L^\infty(0,T;L^2(\Omega)).$$

Define

$$B := \{w \in L^p(0,T;L^p(\Omega)) : \ |w|_{L^p(0,T;L^p(\Omega))} \le C\},$$

where C is the same as in (3.1.4). Then we see that the map T maps B into itself.

Now we want to show that $T(B)$ is relatively compact in B. From the equation in (3.1.3) we have

$$|\langle u_t, v\rangle| \le \tilde{C}\left(\int_\Omega |\nabla u|^{(p-1)q} dx\right)^{\frac{1}{q}}\left(\int_\Omega |\nabla v|^p dx\right)^{\frac{1}{p}} + |f|_{-1,q}\|\nabla v\|_p$$

(here we made use of (3.1.1) and Hölder's inequality). Hence, it holds that

$$|u_t|_{-1,q} \le \tilde{C}\|\nabla u\|_p^{p-1} + |f|_{-1,q}.$$

Therefore, $u_t \in L^q(0,T;W^{-1,q}(\Omega)) \subset L^q(0,T;W^{-1,q}(\Omega)+L^p(\Omega))$, $W^{-1,q}(\Omega)+L^p(\Omega)$ denotes a dual space to $W_0^{1,p}(\Omega) \cap L^q(\Omega)$. We have that

$$W_0^{1,p}(\Omega) \subset L^p(\Omega) \subset W^{-1,q}(\Omega) + L^p(\Omega),$$

where the first embedding is compact (see [28]). Then by Aubin-Lions lemma

$$W := \{v \in L^p(0,T;W_0^{1,p}(\Omega)), \ v_t \in L^q(0,T;W^{-1,q}(\Omega))\}$$

is compact in $L^p(0,T;L^p(\Omega))$. Thus $T(B)$ is relatively compact in B.

It remains to prove that T is continuous. Let us consider a sequence w_n such that

$$w_n \to w \text{ in } B,$$

then for a subsequence

$$w_n(\cdot,t) \to w(\cdot,t) \text{ in } L^p(\Omega) \text{ a.e. } t \in (0,T),$$

and

$$a(l(w_n)) \to a(l(w)) \text{ for a.e. } t \in (0,T).$$

Hence, due to (3.1.1)

$$a(l(w_n)) \to a(l(w)) \text{ in } L^p(0,T).$$

Moreover, for $u_n = T(w_n)$ it holds that

$$u_n \rightharpoonup u_\infty \text{ in } L^p(0,T;W_0^{1,p}(\Omega)),$$

$$u_n \to u_\infty \text{ in } L^p(0,T;L^p(\Omega)),$$

$$u_n(\cdot,t) \rightharpoonup u_\infty(\cdot,t) \text{ in } L^2(\Omega) \text{ a.e. } t \in (0,T),$$

$$\nabla \cdot |\nabla u_n|^{p-2}\nabla u_n \rightharpoonup \chi \text{ in } L^q(0,T;W^{-1,q}(\Omega)),$$

since (3.1.4), (3.1.5) can be obtained in the same way also for u_n. Then for $\varphi \in \mathcal{D}(0,T)$, $v \in W_0^{1,p}(\Omega)$ we have

$$-\int_0^T\int_\Omega u_n v\varphi' dxdt + \int_0^T\int_\Omega a(l(w_n))|\nabla u_n|^{p-2}\nabla u_n \nabla v\varphi dxdt = \int_0^T \langle f,v\rangle\varphi dt. \tag{3.1.6}$$

Passing to the limit above we obtain

$$-\int_0^T\int_\Omega u_\infty v\varphi' dxdt - \int_0^T\int_\Omega a(l(w))\chi v\varphi dxdt = \int_0^T \langle f,v\rangle\varphi dt.$$

Then we have in a sense of distributions

$$(u_\infty)_t - a(l(w))\chi = f, \tag{3.1.7}$$

therefore we see that $(u_\infty)_t \in L^q(0,T;W^{-1,q}(\Omega))$. It remains to prove just that $\chi = \nabla\cdot|\nabla u_\infty|^{p-2}\nabla u_\infty$. From (3.1.6) we have that

$$\langle (u_n)_t, v\rangle + \int_\Omega a(l(w_n))|\nabla u_n|^{p-2}\nabla u_n\nabla v dx = \langle f,v\rangle \text{ in } \mathcal{D}'(0,T). \tag{3.1.8}$$

Taking $v = u_n$ in (3.1.8) we get

$$\frac{1}{2}\frac{d}{dt}|u_n|_2^2 + \int_\Omega a(l(w_n))|\nabla u_n|^p dx = \langle f,u_n\rangle.$$

After an integration on $(0,T)$ we derive

$$\int_0^T a(l(w_n))\int_\Omega |\nabla u_n|^p dxdt = \int_0^T \langle f,u_n\rangle dt + \frac{|u_0|_2^2}{2} - \frac{|u_n(T)|_2^2}{2}.$$

Note that since $u_n, u_\infty \in W\cap L^\infty(0,T;L^2(\Omega))$ then $u_n, u_\infty \in C([0,T];L^r(\Omega))$, $r = \min\{2,p\}$ (see [38]), hence $u_\infty(0) = u_0$. Now passing to the limit we see that

$$\overline{\lim_{n\to\infty}}\int_0^T a(l(w_n))\int_\Omega |\nabla u_n|^p dxdt \le \int_0^T \langle f,u_\infty\rangle dt + \frac{|u_\infty(0)|_2^2}{2} - \frac{|u_\infty(T)|_2^2}{2}.$$

Thus from the inequality

$$\int_0^T a(l(w_n))\int_\Omega \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla v|^{p-2}\nabla v\big)\nabla(u_n - v)dxdt \ge 0$$

taking $\overline{\lim}$ we derive

$$\int_0^T \langle f,u_\infty\rangle dt + \frac{|u_\infty(0)|_2^2}{2} - \frac{|u_\infty(T)|_2^2}{2} + \int_0^T a(l(w))\int_\Omega \chi v dxdt \\ - \int_0^T a(l(w))\int_\Omega |\nabla v|^{p-2}\nabla v\nabla(u_\infty - v)dxdt \ge 0. \tag{3.1.9}$$

From (3.1.7) after having multiplied it by u_∞ and integrated over $Q_T = \Omega \times (0,T)$ we get

$$\int_0^T \langle f, u_\infty \rangle dt + \frac{|u_\infty(0)|_2^2}{2} - \frac{|u_\infty(T)|_2^2}{2} = -\int_0^T a(l(w)) \int_\Omega \chi u_\infty dxdt. \tag{3.1.10}$$

Then combining (3.1.9), (3.1.10) we come to

$$\int_0^T a(l(w)) \int_\Omega \big(-\chi + \nabla \cdot |\nabla v|^{p-2} \nabla v \big)(u_\infty - v) dxdt \geq 0.$$

Taking $v = u_\infty + \eta z$, $\eta > 0$ we obtain for arbitrary $z \in L^p(0,T; W_0^{1,p}(\Omega))$

$$\int_0^T a(l(w)) \int_\Omega \big(-\chi + \nabla \cdot |\nabla (u_\infty + \eta z)|^{p-2} \nabla (u_\infty + \eta z) \big) z dxdt = 0.$$

Letting $\eta \to 0$ we get

$$\int_0^T a(l(w)) \int_\Omega \big(-\chi + \nabla \cdot |\nabla u_\infty|^{p-2} \nabla u_\infty \big) z dxdt \geq 0 \quad \forall z \in L^p(0,T; W_0^{1,p}(\Omega)).$$

And now since $a > 0$ we have

$$\chi = \nabla \cdot |\nabla u_\infty|^{p-2} \nabla u_\infty,$$

which completes the proof of the theorem. □

Theorem 3.1.2. *Let the assumption* (3.1.1) *holds. Suppose in addition that*

$$g \in L^2(\Omega) \tag{3.1.11}$$

and exists a constant L such that

$$|a(\xi) - a(\xi')| \leq L|\xi - \xi'| \quad \forall \xi, \xi' \in \mathbb{R}, \tag{3.1.12}$$

then the solution to (3.1.2) *is unique.*

Proof. Let u_1, u_2 be two solutions to (3.1.2). By subtraction we obtain

$$\langle (u_1 - u_2)_t, v \rangle + a(l(u_1)) \int_\Omega \big(|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2 \big) \nabla v dx$$
$$= (a(l(u_2)) - a(l(u_1))) \int_\Omega |\nabla u_2|^{p-2} \nabla u_2 \nabla v dx.$$

Taking $v = u_1 - u_2$ and using (3.1.1), (3.1.12) we get

$$\frac{1}{2} \frac{d}{dt} |u_1 - u_2|_2^2 + \lambda \int_\Omega \big(|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2 \big) \cdot \nabla (u_1 - u_2) dx$$
$$\leq L \left| \int_\Omega g(u_1 - u_2) dx \right| \left| \int_\Omega |\nabla u_2|^{p-2} \nabla u_2 \nabla (u_1 - u_2) dx \right|. \tag{3.1.13}$$

From Hölder's inequality we derive

$$\begin{aligned}
\left|\int_\Omega |\nabla u_2|^{p-2}\nabla u_2 \nabla(u_1-u_2)dx\right| &\le \int_\Omega |\nabla u_2|^{p-1}|\nabla(u_1-u_2)|dx \\
&\le \int_\Omega \big(|\nabla u_1|+|\nabla u_2|\big)^{p-1}|\nabla(u_1-u_2)|dx \\
&= \int_\Omega \big(|\nabla u_1|+|\nabla u_2|\big)^{\frac{p}{2}}\big(|\nabla u_1|+|\nabla u_2|\big)^{\frac{p}{2}-1}|\nabla(u_1-u_2)|dx \\
&\le \left(\int_\Omega \big(|\nabla u_1|+|\nabla u_2|\big)^p dx\right)^{\frac{1}{2}}\left(\int_\Omega \big(|\nabla u_1|+|\nabla u_2|\big)^{p-2}|\nabla(u_1-u_2)|^2 dx\right)^{\frac{1}{2}}.
\end{aligned}$$

From Lemma A.1 we obtain

$$\begin{aligned}
&\int_\Omega (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)\cdot\nabla(u_1-u_2)dx \\
&\qquad\qquad \ge c_p \int_\Omega \big(|\nabla u_1|+|\nabla u_2|\big)^{p-2}|\nabla(u_1-u_2)|^2 dx.
\end{aligned}$$

Combining (3.1.13) and two inequalities above leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2 + \lambda c_p \int_\Omega \big(|\nabla u_1|+|\nabla u_2|\big)^{p-2}|\nabla(u_1-u_2)|^2 dx \\
&\le L|g|_2|u_1-u_2|_2 \left(\int_\Omega \big(|\nabla u_1|+|\nabla u_2|\big)^p dx\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_\Omega \big(|\nabla u_1|+|\nabla u_2|\big)^{p-2}|\nabla(u_1-u_2)|^2 dx\right)^{\frac{1}{2}} \\
&\le \frac{\lambda c_p}{2}\int_\Omega \big(|\nabla u_1|+|\nabla u_2|\big)^{p-2}|\nabla(u_1-u_2)|^2 dx + C(t)|u_1-u_2|_2^2.
\end{aligned}$$

(In the last inequality above we use Young's inequality and $C \in L^1(0,T)$). Therefore, we have

$$\frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2 \le C(t)|u_1-u_2|_2^2.$$

The uniqueness follows then from Gronwall's inequality. □

3.2 Stationary problem

The associated stationary problem to the problem (3.0.1) is the following problem

$$\begin{cases} -\nabla\cdot a(l(u))|\nabla u|^{p-2}\nabla u = f \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma, \end{cases} \tag{3.2.1}$$

where $l(u)$ is defined by

$$l(u) = \int_\Omega g(x)u(x)dx,\ g \in L^q(\Omega). \tag{3.2.2}$$

The weak formulation of the problem (3.2.1) can be written as follows

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega a(l(u))|\nabla u|^{p-2}\nabla u \nabla v dx = \langle f, v\rangle \quad \forall v \in W_0^{1,p}(\Omega). \end{cases} \tag{3.2.3}$$

By φ we denote the solution to

$$\begin{cases} \varphi \in W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega |\nabla \varphi|^{p-2}\nabla \varphi \nabla v dx = \langle f, v\rangle \quad \forall v \in W_0^{1,p}(\Omega). \end{cases} \tag{3.2.4}$$

It is known that for $f \in W^{-1,q}(\Omega)$ (3.2.4) admits a unique solution [14].

Theorem 3.2.1. *Suppose that* (3.0.3) *holds,* $1 < p < +\infty$. *Then for* $f \in W^{-1,q}(\Omega)$, *the mapping* $u \mapsto l(u)$ *is one-to-one mapping from the set of solutions to* (3.2.3) *onto the set of solutions in* $\mathbb{R}$ *of the equation*

$$a(\mu)^{\frac{1}{p-1}}\mu = l(\varphi). \tag{3.2.5}$$

Proof. Let u_∞ be the solution to the stationary problem, then

$$\begin{aligned} \int_\Omega a(l(u_\infty))|\nabla u_\infty|^{p-2}\nabla u_\infty \nabla v dx &= \langle f, v\rangle \\ &= \int_\Omega |\nabla \varphi|^{p-2}\nabla \varphi \nabla v dx \quad \forall v \in W_0^{1,p}(\Omega), \end{aligned} \tag{3.2.6}$$

which implies

$$a(l(u_\infty))^{\frac{1}{p-1}}u_\infty = \varphi. \tag{3.2.7}$$

Multiplying by g and integrating over Ω we get

$$a(l(u_\infty))^{\frac{1}{p-1}}l(u_\infty) = l(\varphi), \tag{3.2.8}$$

which implies that $l(u_\infty)$ is a solution to (3.2.5).

Conversely, let μ be a solution to (3.2.5). There exists a unique solution u to

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega a(\mu)|\nabla u|^{p-2}\nabla u \nabla v dx = \langle f, v\rangle \quad \forall v \in W_0^{1,p}(\Omega). \end{cases} \tag{3.2.9}$$

From (3.2.4), (3.2.9) we obtain

$$a(\mu)^{\frac{1}{p-1}}u = \varphi.$$

Applying l we get

$$a(\mu)^{\frac{1}{p-1}} l(u) = l(\varphi) = a(\mu)^{\frac{1}{p-1}} \mu.$$

Since $a(\mu) > 0$ we have $l(u) = \mu$.

Now to show injectivity we have

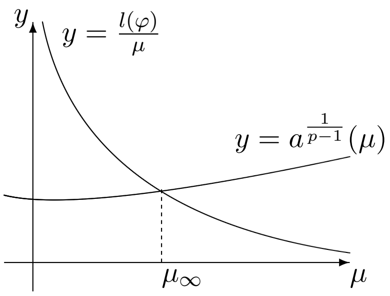
$$l(u_1) = l(u_2) \;\Rightarrow\; a(l(u_1)) = a(l(u_2)) \;\Rightarrow\; u_1 = u_2.$$

We used the uniqueness of the solution of (3.2.9). $\square$

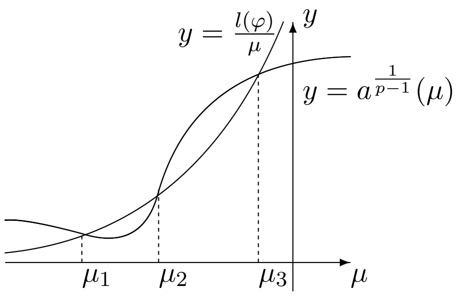
When $l(\varphi) = 0$ the only solution to (3.2.5) is given by $\mu = 0$ and the stationary problem (3.2.3) has a unique solution given by

$$u = \varphi / a^{\frac{1}{p-1}}(0). \tag{3.2.10}$$

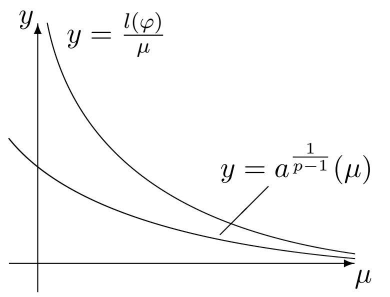
When $l(\varphi) \neq 0$, then either $l(\varphi) > 0$ or $l(\varphi) < 0$. When $l(\varphi) \neq 0$ then the solutions to (3.2.3) are the points where the graph of $a^{\frac{1}{p-1}}$ intersects the hyperbola $\mu \to l(\varphi)/\mu$. One should note that (3.2.3) can admit no solution, several, infinitely many. See some examples on Figure 3.2.1.



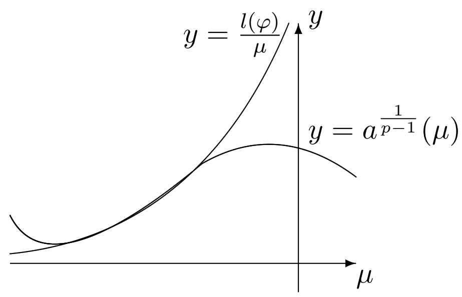
(a) The case $l(\varphi) > 0$, a unique solution



(b) The case $l(\varphi) < 0$, several solutions



(c) The case $l(\varphi) > 0$, no solution



(d) The case $l(\varphi) < 0$, infinitely many solutions

Figure 3.2.1

3.3 Asymptotic behaviour in a case of a single equilibrium

We will suppose $l(\varphi) > 0$ (the same could be done for $l(\varphi) < 0$). In this section we assume that (3.2.5) admits a unique solution μ_∞ and we denote then by u_∞ the unique solution to (3.2.3).

Lemma 3.3.1. *Let u be a weak solution to* (3.0.1). *If $1 < p < +\infty$ for $n = 1$, $\frac{2n}{2+n} < p < +\infty$ for $n \geq 2$ and suppose that* (3.0.4), (3.1.1) *hold, then $u(t)$ is uniformly bounded in $L^2(\Omega)$.*

Proof. Recall that for u solution of (3.0.1) we have

$$\langle u_t, v\rangle + \int_\Omega a(l(u(t)))|\nabla u|^{p-2}\nabla u \nabla v dx = \langle f, v\rangle \quad \forall v \in W_0^{1,p}(\Omega).$$

Taking $v = u$ and using (3.1.1) we get

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + \lambda \int_\Omega |\nabla u|^p dx \leq \langle f, u\rangle.$$

Using Hölder's, Poincaré's and Young's inequalities we obtain

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + \lambda \int_\Omega |\nabla u|^p dx \leq |f|_{-1,q}\|\nabla u\|_p \leq \frac{1}{\lambda^{q-1}q}|f|_{-1,q}^q + \frac{\lambda}{p}\|\nabla u\|_p^p.$$

It implies

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + \frac{\lambda}{q} \int_\Omega |\nabla u|^p dx \leq \frac{1}{\lambda^{q-1}q}|f|_{-1,q}^q. \tag{3.3.1}$$

Due to the fact that if $n = 1$ for $1 < p < \infty$ or if $n \geq 2$ for $\frac{2n}{2+n} < p < +\infty$ it holds that $W_0^{1,p}(\Omega) \subset L^2(\Omega)$ (see [28]), therefore for some $C > 0$

$$|u|_2 \leq C\|\nabla u\|_p.$$

Hence, from (3.3.1) we can deduce

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + \frac{\lambda C'}{q}|u|_2^p \leq \frac{1}{\lambda^{q-1}q}|f|_{-1,q}^q.$$

Thus by Lemma A.6 with $y(t) = |u(t)|_2^2$ we get

$$|u|_2 \leq \max\left\{|u_0|_2, \frac{|f|_{-1,q}^{q-1}}{\lambda^{q-1}C'^{\frac{1}{p}}}\right\},$$

which completes the proof of the lemma. □

Let us assume that (3.1.11) holds and

$$f, g \geq 0, \quad f, g \not\equiv 0, \quad g \in L^2(\Omega). \tag{3.3.2}$$

We set

$$l_0 = \liminf_{t\to\infty} l(u(t)), \quad L_0 = \limsup_{t\to\infty} l(u(t)), \tag{3.3.3}$$

From Lemma 3.3.1 it follows that l_0, L_0 both are finite. Then we can define

$$m_0 = \inf_{[l_0,L_0]} a(\xi), \quad M_0 = \sup_{[l_0,L_0]} a(\xi). \tag{3.3.4}$$

Lemma 3.3.2. *Let $p \geq 2$, then it holds that*

$$\frac{l(\varphi)}{M_0^{\frac{1}{p-1}}} \leq l_0 \leq L_0 \leq \frac{l(\varphi)}{m_0^{\frac{1}{p-1}}}, \tag{3.3.5}$$

Proof. By the definition of lim inf and lim sup we have for $\varepsilon > 0$ there exists $t_0 = t_0(\varepsilon)$ such that

$$l_0 - \varepsilon \leq l(u(t)) \leq L_0 + \varepsilon \quad \forall t \geq t_0$$

and for $\delta = \delta(\varepsilon)$ such that $\lim\limits_{\varepsilon\to 0} \delta(\varepsilon) = 0$ it holds

$$m_0 - \delta \leq a(l(u(t))) \leq M_0 + \delta \quad \forall t \geq t_0.$$

From (3.0.1), (3.2.4) we derive

$$\langle u_t, v\rangle + \int_\Omega a(l(u(t)))|\nabla u|^{p-2}\nabla u \nabla v dx = \langle f, v\rangle$$
$$= A^{p-1}\int_\Omega \left|\frac{\nabla\varphi}{A}\right|^{p-2} \frac{\nabla\varphi}{A}\nabla v dx \quad \forall v \in W_0^{1,p}(\Omega).$$

Choosing $A = (M_0+\delta)^{\frac{1}{p-1}}$ we obtain

$$\left\langle \left(u - \frac{\varphi}{A}\right)_t, v\right\rangle + \int_\Omega a(l(u(t)))\left(|\nabla u|^{p-2}\nabla u - \left|\frac{\nabla\varphi}{A}\right|^{p-2}\frac{\nabla\varphi}{A}\right)\nabla v dx$$
$$= \left(A^{p-1} - a(l(u(t)))\right)\int_\Omega \left|\frac{\nabla\varphi}{A}\right|^{p-2}\frac{\nabla\varphi}{A}\nabla v dx \quad \forall v \in W_0^{1,p}(\Omega). \tag{3.3.6}$$

For $t \geq t_0$ we have $A^{p-1} - a(l(u(t))) = M_0 + \delta - a(l(u(t))) \geq 0$ and for $v \leq 0$ we get

$$\int_\Omega \left|\frac{\nabla\varphi}{A}\right|^{p-2}\frac{\nabla\varphi}{A}\nabla v dx = \langle f, v\rangle \leq 0.$$

Thus, taking $v = -\left(u - \frac{\varphi}{A}\right)^-$ in (3.3.6) we obtain for $t \geq t_0$

$$\frac{d}{dt}\left|\left(u - \frac{\varphi}{A}\right)^-\right|_2^2$$
$$+ \int_\Omega a(l(u(t)))\left(|\nabla u|^{p-2}\nabla u - \left|\frac{\nabla\varphi}{A}\right|^{p-2}\frac{\nabla\varphi}{A}\right)\nabla\left(u - \frac{\varphi}{A}\right)^- dx \leq 0. \tag{3.3.7}$$

If $p \geq 2$, then Lemma A.2 and (3.1.1) imply

$$\frac{d}{dt}\left|\left(u-\frac{\varphi}{A}\right)^{-}\right|_2^2 + \lambda C \int_\Omega \left|\nabla\left(u-\frac{\varphi}{A}\right)^{-}\right|^p dx \leq 0.$$

Taking into account that for $p \geq 2$ it holds that

$$|u|_2^p \leq C|\nabla u|_p^p, \tag{3.3.8}$$

we derive

$$\frac{d}{dt}\left|\left(u-\frac{\varphi}{A}\right)^{-}\right|_2^2 + \lambda C' \left|\left(u-\frac{\varphi}{A}\right)^{-}\right|_2^p \leq 0 \quad \forall t \geq t_0. \tag{3.3.9}$$

Then by Lemma A.6, (3.3.9) with $y(t) = |(u-\dfrac{\varphi}{A})^{-}|_2^2$ it follows that

$$\varepsilon_0(t) := \left(u-\frac{\varphi}{A}\right)^{-} \to 0 \text{ as } t \to +\infty \text{ in } L^2(\Omega).$$

Writing that

$$u-\frac{\varphi}{A} = \left(u-\frac{\varphi}{A}\right)^{+} - \left(u-\frac{\varphi}{A}\right)^{-}$$

we get

$$\frac{\varphi}{(M_0+\delta)^{\frac{1}{p-1}}} - \varepsilon_0(t) \leq u. \tag{3.3.10}$$

Similarly as above assuming ε small enough, i.e. such that $m_0 - \delta > 0$, choosing $A = (m_0-\delta)^{\frac{1}{p-1}}$ and $v = \left(u-\dfrac{\varphi}{A}\right)^{+}$ in (3.3.6) we can show that

$$\varepsilon_1(t) := \left(u-\frac{\varphi}{A}\right)^{+} \to 0 \text{ as } t \to \infty \text{ in } L^2(\Omega)$$

and

$$u \leq \frac{\varphi}{(m_0-\delta)^{\frac{1}{p-1}}} + \varepsilon_1(t). \tag{3.3.11}$$

Combining (3.3.10) and (3.3.11) we have

$$\frac{\varphi}{(M_0+\delta)^{\frac{1}{p-1}}} - \varepsilon_0(t) \leq u(t) \leq \frac{\varphi}{(m_0-\delta)^{\frac{1}{p-1}}} + \varepsilon_1(t) \quad t \geq t_0.$$

Since $g \geq 0$ it comes

$$\frac{l(\varphi)}{(M_0+\delta)^{\frac{1}{p-1}}} - l(\varepsilon_0(t)) \leq l(u(t)) \leq \frac{l(\varphi)}{(m_0-\delta)^{\frac{1}{p-1}}} + l(\varepsilon_1(t)) \quad t \geq t_0. \tag{3.3.12}$$

Taking $t \to \infty$ we obtain

$$\frac{l(\varphi)}{(M_0+\delta)^{\frac{1}{p-1}}} \leq l_0 \leq L_0 \leq \frac{l(\varphi)}{(m_0-\delta)^{\frac{1}{p-1}}} \quad t \geq t_0.$$

Then letting $\varepsilon \to 0$ we get the statement of the lemma. □

Lemma 3.3.3. *Let now $1 < p < 2$ for $n = 1$ or for $n \geq 2$ and $\frac{2n}{2+n} < p < 2$ and in addition $f \in L^2(\Omega), u_0 \in W_0^{1,p}(\Omega)$. Then (3.3.5) holds true.*

Proof. Multiplying the equation in (3.0.1) by u_t and integrating over Ω we get

$$\int_\Omega u_t^2 dx + a\left(l(u(t))\right) \int_\Omega |\nabla u|^{p-2} \nabla u \nabla u_t dx = \int_\Omega f u_t dx.$$

Remark that

$$\frac{1}{p}\frac{d}{dt}\|\nabla u\|_p^p = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla u_t dx.$$

Hence, we can write using Hölder's and Young's inequalities

$$\int_\Omega u_t^2 dx + a\left(l(u(t))\right) \frac{1}{p}\frac{d}{dt}\|\nabla u\|_p^p \leq \left|\int_\Omega f u_t dx\right| \leq |f|_2 |u_t|_2 \leq \frac{|f|_2^2}{2} + \frac{|u_t|_2^2}{2}.$$

Therefore, we have

$$\frac{|u_t|_2^2}{2} + a\left(l(u(t))\right) \frac{1}{p}\frac{d}{dt}\|\nabla u\|_p^p \leq \frac{|f|_2^2}{2}.$$

Since the first term is nonnegative, using (3.1.1) we obtain by integration

$$\|\nabla u\|_p^p \leq \|\nabla u_0\|_p^p + \frac{p|f|_2^2 t}{2\lambda} \tag{3.3.13}$$

From Lemma A.1 we get

$$\begin{aligned}
&\int_\Omega a(l(u(t))) \left(|\nabla u|^{p-2}\nabla u - \left|\frac{\nabla\varphi}{A}\right|^{p-2}\frac{\nabla\varphi}{A}\right) \nabla\left(u - \frac{\varphi}{A}\right)^- dx \\
&\qquad \geq \lambda c_p \int_\Omega \left(|\nabla u| + \left|\frac{\nabla\varphi}{A}\right|\right)^{p-2} \left|\nabla\left(u - \frac{\varphi}{A}\right)^-\right|^2 dx \\
&\qquad \geq \lambda c_p \frac{\left(\int_\Omega \left|\nabla\left(u - \frac{\varphi}{A}\right)^-\right|^p dx\right)^{\frac{2}{p}}}{\left(\int_\Omega \left(|\nabla u| + \left|\frac{\nabla\varphi}{A}\right|\right)^p\right)^{\frac{2-p}{p}}}
\end{aligned} \tag{3.3.14}$$

Here we made use of Hölder's inequality with exponents $\frac{2}{p}$, $\frac{2}{2-p}$. As above we can come to the inequality (3.3.7). Due to the embedding $W_0^{1,p}(\Omega) \subset L^2(\Omega)$ for $n = 1$ and $1 < p < \infty$ or for $n \geq 2$ and $\frac{2n}{2+n} < p < +\infty$ [28] the (3.3.8) holds and then combining (3.3.7), (3.3.13) and (3.3.14) we derive

$$\frac{d}{dt}\left|\left(u - \frac{\varphi}{A}\right)^-\right|_2^2 + C(1+t)^{\frac{p-2}{p}} \left|\left(u - \frac{\varphi}{A}\right)^-\right|_2^2 \leq 0 \quad \forall t \geq t_0. \tag{3.3.15}$$

Multiplying by $\exp(C\int_0^t (1+s)^{\frac{p-2}{p}} ds)$ and denoting $y(t) = \left|\left(u - \frac{\varphi}{A}\right)^-\right|_2^2$ we get

$$y(t) \leq y(0) e^{-C((t+1)^{\frac{2(p-1)}{p}} - 1)}.$$

Hence, $\varepsilon_0(t) \to 0$ as $t \to +\infty$ in $L^2(\Omega)$. The same can be shown for $\varepsilon_1(t)$ and we can complete the proof as in the previous lemma. □

Lemma 3.3.4. *Under our assumptions it holds that*

$$l_0 < \mu_\infty < L_0 \text{ or } l_0 = \mu_\infty = L_0. \tag{3.3.16}$$

Proof. Since the equation (3.2.5) has a unique solution it holds that

$$a(\mu)^{\frac{1}{p-1}} < \frac{l(\varphi)}{\mu} \quad 0 < \mu < \mu_\infty, \tag{3.3.17}$$

$$a(\mu)^{\frac{1}{p-1}} > \frac{l(\varphi)}{\mu} \quad \mu > \mu_\infty. \tag{3.3.18}$$

Let us first consider the case $l_0 < L_0$. Suppose for instance that

$$\mu_\infty \leq l_0 < L_0. \tag{3.3.19}$$

Denote ξ_0 a point such that $\xi_0 \in [l_0, L_0]$, $a(\xi_0) = m_0$.

- If $\xi_0 = \mu_\infty$ then from (3.3.5) we get

$$\mu_\infty \leq l_0 < L_0 \leq \frac{l(\varphi)}{a(\xi_0)^{\frac{1}{p-1}}} = \frac{l(\varphi)}{a(\mu_\infty)^{\frac{1}{p-1}}} = \mu_\infty,$$

  which is impossible.

- If $\xi_0 \neq \mu_\infty$ then from (3.3.5) and (3.3.18)

$$\mu_\infty \leq l_0 < L_0 \leq \frac{l(\varphi)}{a(\xi_0)^{\frac{1}{p-1}}} < \xi_0,$$

  what contradicts $\xi_0 \in [l_0, L_0]$.

Hence (3.3.19) cannot occur.

Let us suppose

$$l_0 < L_0 \leq \mu_\infty \tag{3.3.20}$$

and $\zeta_0 \in [l_0, L_0]$ be such that $a(\zeta_0) = M_0$.

- If $\zeta_0 = \mu_\infty$ then from (3.3.5) we have

$$\mu_\infty \geq l_0 > L_0 \geq \frac{l(\varphi)}{a(\zeta_0)^{\frac{1}{p-1}}} = \frac{l(\varphi)}{a(\mu_\infty)^{\frac{1}{p-1}}} = \mu_\infty.$$

  We came to a contradiction.

- If $\zeta_0 \neq \mu_\infty$ then from (3.3.5) and (3.3.17) follow

$$\mu_\infty \geq l_0 > L_0 \geq \frac{l(\varphi)}{a(\zeta_0)^{\frac{1}{p-1}}} > \zeta_0.$$

But this cannot happen, since $\zeta_0 \in [l_0, L_0]$.

Thus, if $l_0 < L_0$ then $l_0 < \mu_\infty < L_0$.

Suppose now that $l_0 = L_0$. Then (3.3.5) implies

$$\frac{l(\varphi)}{a(l_0)^{\frac{1}{p-1}}} \leq l_0 \leq L_0 \leq \frac{l(\varphi)}{a(l_0)^{\frac{1}{p-1}}}.$$

It means that l_0 is a solution to the equation (3.2.5). □

Theorem 3.3.5. *Suppose we are under the conditions of the preceding lemmas in case $p \geq 2$ with in addition*

$$a(\mu)^{\frac{1}{p-1}} \geq a(\mu_\infty)^{\frac{1}{p-1}} \quad \forall\, 0 < \mu < \mu_\infty \tag{3.3.21}$$

or

$$a(\mu)^{\frac{1}{p-1}} \leq a(\mu_\infty)^{\frac{1}{p-1}} \quad \forall \mu > \mu_\infty \tag{3.3.22}$$

or

$$\begin{aligned} a(\mu)^{\frac{1}{p-1}} &< \frac{l(\varphi)}{2\mu_\infty - \mu} \quad \forall \mu \in (\mu_\infty, 2\mu_\infty), \\ a(\mu)^{\frac{1}{p-1}} &> \frac{l(\varphi)}{2\mu_\infty - \mu} \quad \forall \mu \in (0, \mu_\infty), \end{aligned} \tag{3.3.23}$$

then it holds that

$$\lim_{t\to\infty} u(t) = u_\infty \textit{ in } L^2(\Omega). \tag{3.3.24}$$

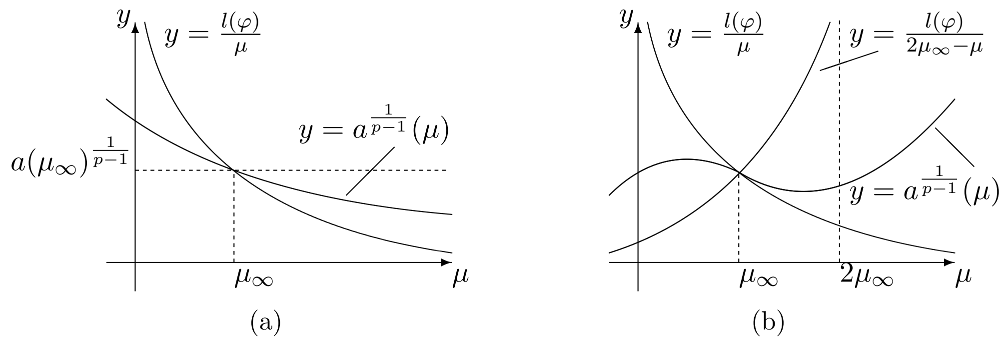


Figure 3.3.2

Proof. Let us show the impossibility of

$$l_0 < \mu_\infty < L_0.$$

Suppose that we are in the case of (3.3.21) (see Figure 3.3.2a). Then taking into account (3.3.18) we have

$$m_0^{\frac{1}{p-1}} = \inf_{[l_0,L_0]} (a(\mu))^{\frac{1}{p-1}} = \inf_{[\mu_\infty,L_0]} (a(\mu))^{\frac{1}{p-1}} > \frac{l(\varphi)}{L_0},$$

which contradicts (3.3.5).

Let now (3.3.22) is true (Figure 3.3.2a). Then by (3.3.17) follows

$$M_0^{\frac{1}{p-1}} = \sup_{[l_0,L_0]} (a(\mu))^{\frac{1}{p-1}} = \sup_{[l_0,\mu_\infty]} (a(\mu))^{\frac{1}{p-1}} < \frac{l(\varphi)}{l_0},$$

which is impossible due to (3.3.5).

Consider case (3.3.23) (see Figure 3.3.2b). Suppose first that

$$L_0 - \mu_\infty \geq \mu_\infty - l_0.$$

Then, (3.3.18), (3.3.23) imply

$$m_0^{\frac{1}{p-1}} = \inf_{[l_0,L_0]} (a(\mu))^{\frac{1}{p-1}} > \frac{l(\varphi)}{L_0},$$

what cannot occur due to (3.3.5). From another side, if

$$L_0 - \mu_\infty < \mu_\infty - l_0,$$

then from (3.3.17) and (3.3.23) we can conclude

$$M_0^{\frac{1}{p-1}} = \sup_{[l_0,L_0]} (a(\mu))^{\frac{1}{p-1}} < \frac{l(\varphi)}{l_0},$$

which again contradicts (3.3.5).

Hence, $l_0 < \mu_\infty < L_0$ cannot be satisfied and by (3.3.16) follows that $l_0 = \mu_\infty = L_0$, which implies

$$\lim_{t\to\infty} a(l(u(t))) = a(\mu_\infty). \tag{3.3.25}$$

From (3.1.2), (3.2.3) and since u_∞ is independent on t we get

$$\begin{aligned}\langle (u-u_\infty)_t, v\rangle &+ a\big(l(u(t))\big)\int_\Omega \big(|\nabla u|^{p-2}\nabla u - |\nabla u_\infty|^{p-2}\nabla u_\infty\big)\nabla v dx \\ &= \Big(a(\mu_\infty) - a\big(l(u(t))\big)\Big)\int_\Omega |\nabla u_\infty|^{p-2}\nabla u_\infty \nabla v dx \quad \forall v \in W_0^{1,p}(\Omega). \end{aligned} \tag{3.3.26}$$

Taking $v = u - u_\infty$ in (3.3.26), by Lemma A.2 for $p \geq 2$, (3.1.1) and using Hölder's and Young's inequalities we derive for some constant C

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&|u-u_\infty|_2^2 + \lambda C|\nabla(u-u_\infty)|_p^p \\ &\leq \varepsilon|\nabla(u-u_\infty)|_p^p + C(\varepsilon)\int_\Omega |a(\mu_\infty) - a(l(u(t))|^q|\nabla u_\infty|^p dx. \end{aligned} \tag{3.3.27}$$

We choose $\varepsilon = \frac{\lambda C}{2}$ and since $W_0^{1,p} \subset L^2(\Omega)$ for $p \geq 2$ we get for some constants

$$\frac{d}{dt}|u-u_\infty|_2^2 + C|u-u_\infty|_2^p \\ \leq \widetilde{C}\int_\Omega |a(\mu_\infty) - a(l(u(t))|^q |\nabla u_\infty|^p dx = \varepsilon(t). \quad (3.3.28)$$

By (3.3.25) and since $u_\infty \in W_0^{1,p}(\Omega)$ one has $\lim_{t\to\infty} \varepsilon(t) = 0$. The result follows from Lemma A.6 with $y(t) = |u-u_\infty|_2^2$. □

Theorem 3.3.6. *Let the assumptions of Lemma 3.3.3 and Theorem 3.3.5 hold. Then*

$$|u-u_\infty|_2^2 \leq |u_0-u_\infty|_2^2 e^{-C_1((t+1)^{\frac{2(p-1)}{p}}-1)} + C_2\int_0^t \big|a(\mu_\infty)-a(l(u(s))\big|^2 e^{-C_1\int_s^t (1+\tau)^{\frac{p-2}{p}}d\tau} ds,$$

where $\lim_{t\to\infty} a(l(u(t))) = a(\mu_\infty)$.

Proof. Let now $\frac{2n}{2+n} < p < 2$ if $n \geq 2$ and $1 < p < 2$ for $n = 1$. Then since in this case we still have $W_0^{1,p}(\Omega) \subset L^2(\Omega)$, from Hölder's inequality one can derive (see (3.3.13), (3.3.14)) for some constant C that

$$\int_\Omega \big(|\nabla u_\infty| + |\nabla u|\big)^{p-2} |\nabla(u-u_\infty)|^2 dx \\ \geq C(1+t)^{\frac{p-2}{p}} |\nabla(u-u_\infty)|_p^2 \geq \tilde{C}(1+t)^{\frac{p-2}{p}}|u-u_\infty|_2^2. \quad (3.3.29)$$

Remark also that again by Hölder's inequality we have

$$\left|\int_\Omega |\nabla u_\infty|^{p-2}\nabla u_\infty \nabla(u-u_\infty)dx\right| \leq \int_\Omega \big(|\nabla u_\infty| + |\nabla u|\big)^{p-1}|\nabla(u-u_\infty)|dx \\ \leq \left(\int_\Omega \big(|\nabla u_\infty| + |\nabla u|\big)^{p-2}|\nabla(u-u_\infty)|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \big(|\nabla u_\infty| + |\nabla u|\big)^p dx\right)^{\frac{1}{2}}. \quad (3.3.30)$$

Hence, combining the inequality above and Lemma A.1, (3.3.26) with $v = u - u_\infty$ implies

$$\frac{1}{2}\frac{d}{dt}|u-u_\infty|_2^2 + \lambda c_p \int_\Omega \big(|\nabla u_\infty| + |\nabla u|\big)^{p-2}|\nabla(u-u_\infty)|^2 dx \\ \leq C\left(\int_\Omega \big(|\nabla u_\infty| + |\nabla u|\big)^{p-2}|\nabla(u-u_\infty)|^2 dx\right)^{\frac{1}{2}} \big|a(\mu_\infty) - a(l(u(t))\big| \\ \leq \varepsilon \int_\Omega \big(|\nabla u_\infty| + |\nabla u|\big)^{p-2}|\nabla(u-u_\infty)|^2 dx + C(\varepsilon)\big|a(\mu_\infty) - a(l(u(t))\big|^2. \quad (3.3.31)$$

Here we used the Young inequality. Choosing $\varepsilon = \dfrac{\lambda c_p}{2}$ and using (3.3.29) from above we get

$$\frac{1}{2}\frac{d}{dt}|u-u_\infty|_2^2 + C_1(1+t)^{\frac{p-2}{p}}|u-u_\infty|_2^2 \le C_2\big|a(\mu_\infty)-a(l(u(t))\big|^2. \tag{3.3.32}$$

Therefore, by Gronwall's lemma with $y(t) = |u-u_\infty|_2^2$ we obtain

$$y(t) \le y(0)e^{-C_1((t+1)^{\frac{2(p-1)}{p}}-1)} + C_2\int_0^t \big|a(\mu_\infty)-a(l(u(s))\big|^2 e^{-C_1\int_s^t(1+\tau)^{\frac{p-2}{p}}d\tau}ds.$$

Hence, by (3.3.25) we get the result. □

Theorem 3.3.7. *If*

$$\frac{Ll(\varphi)}{\lambda^2(p-1)}\max\{(2\lambda)^{\frac{2-p}{p-1}},(2\Lambda)^{\frac{2-p}{p-1}}\} < 1 \tag{3.3.33}$$

then it holds that

$$\lim_{t\to\infty} u(t) = u_\infty \text{ in } L^2(\Omega). \tag{3.3.34}$$

Proof. Let ξ_0, ζ_0 be such that

$$a(\xi_0) = m_0, \quad a(\zeta_0) = M_0.$$

We claim that the function $a^{\frac{1}{p-1}}$ is locally Lipschitz continuous. Indeed, from Lemma A.3, (3.1.1) and (3.1.12) we get

$$\begin{aligned}|a(\xi)^{\frac{1}{p-1}} - a(\xi')^{\frac{1}{p-1}}| &\le \frac{1}{p-1}|a(\xi)-a(\xi')|(a(\xi)+a(\xi'))^{\frac{2-p}{p-1}} \le \\ &\le L'|\xi-\xi'|,\end{aligned}$$

where $L' = \frac{L}{p-1}\max\{(2\lambda)^{\frac{2-p}{p-1}},(2\Lambda)^{\frac{2-p}{p-1}}\}$. Then from above and (3.3.5) we obtain

$$\begin{aligned}M_0^{\frac{1}{p-1}} - m_0^{\frac{1}{p-1}} &= |a(\zeta_0)^{\frac{1}{p-1}} - a(\xi_0)^{\frac{1}{p-1}}| \le L'|\zeta_0-\xi_0| \le L'(L_0-l_0) \le \\ \le L'\left|\frac{l(\varphi)}{M_0^{\frac{1}{p-1}}} - \frac{l(\varphi)}{m_0^{\frac{1}{p-1}}}\right| &= L'l(\varphi)\frac{M_0^{\frac{1}{p-1}} - m_0^{\frac{1}{p-1}}}{(m_0M_0)^{\frac{1}{p-1}}} \le \frac{L'l(\varphi)}{\lambda^2}(M_0^{\frac{1}{p-1}} - m_0^{\frac{1}{p-1}}).\end{aligned}$$

(3.3.33) implies that $m_0 = M_0$ and from (3.3.5) we can conclude that $l_0 = L_0$. Therefore, we can proceed as in the proof of previous theorem and the result follows. □

3.4 The case when $f = \kappa g$

In this section we consider the problem (3.0.1) assuming that

$$f = \kappa g, \quad \kappa > 0. \tag{3.4.1}$$

Remark that if the condition (3.4.1) holds, than one can introduce the energy functional

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_0^{l(u)} \frac{\kappa}{a(s)}ds, \tag{3.4.2}$$

and its critical points are solutions to the stationary problem (3.2.3). Therefore, we can reproduce what we did in Chapter 2. Recall that the stationary points are determined by the solutions to

$$a(\mu) = l(\varphi)^{p-1}\mu^{1-p} := y(\mu) \tag{3.4.3}$$

(see Theorem 3.2.1).

3.4.1 Remarks on stationary points

Lemma 3.4.1. *Let φ be a solution to* (3.2.4)*. Then it holds that for $\alpha \geq 0$*

(*i*) $\displaystyle \frac{1}{p}\|\nabla\alpha\varphi\|_p^p = \int_0^{l(\alpha\varphi)} \frac{\kappa}{y(s)}ds = \frac{\kappa\alpha^p l(\varphi)}{p}$;

(*ii*) $\displaystyle E(\alpha\varphi) = \kappa\int_0^{l(\alpha\varphi)} \left(\frac{1}{y(s)} - \frac{1}{a(s)}\right) ds.$

Proof. From (3.2.4) with $v = \varphi$ and (3.4.1) we get that

$$\frac{1}{p}\|\nabla\alpha\varphi\|_p^p = \frac{\alpha^p}{p}\int_\Omega |\nabla\varphi|^p dx = \frac{\alpha^p}{p}\int_\Omega f\varphi dx = \frac{\kappa\alpha^p}{p}\int_\Omega g\varphi dx = \frac{\kappa\alpha^p}{p}l(\varphi).$$

Computing

$$\int_0^{l(\alpha\varphi)} \frac{\kappa}{y(s)}ds = \int_0^{l(\alpha\varphi)} \frac{\kappa s^{p-1}}{l(\varphi))^{p-1}}ds = \frac{\kappa l(\alpha\varphi)^p}{pl(\varphi)^{p-1}} = \frac{\kappa\alpha^p}{p}l(\varphi).$$

Thus, (*i*) holds. Let us consider now

$$\begin{aligned} E(\alpha\varphi) &= \frac{1}{p}\|\nabla\alpha\varphi\|_p^p - \int_0^{l(\alpha\varphi)} \frac{\kappa}{a(s)}ds \\ &= \left(\frac{1}{p}\|\nabla\alpha\varphi\|_p^p - \int_0^{l(\alpha\varphi)} \frac{\kappa}{y(s)}ds\right) + \kappa\int_0^{l(\alpha\varphi)} \left(\frac{1}{y(s)} - \frac{1}{a(s)}\right) ds. \end{aligned} \tag{3.4.4}$$

Hence, by (*i*) we see that the first term in (3.4.4) vanishes and the result follows. $\square$

Corollary 3.4.2. *Let u_* be a solution of the problem* (3.2.3) *corresponding to the solutions μ_* of the equation* (3.4.3)*, then by Theorem 3.2.1* $u_* = \dfrac{\varphi}{a(\mu_*)^{\frac{1}{p-1}}}$ *and one has*

(*i*) $\displaystyle \frac{1}{p}\|\nabla u_*\|_p^p = \int_0^{l(u_*)} \frac{\kappa}{y(s)} ds = \frac{\kappa l(u_*)^p}{pl(\varphi)^{p-1}};$

(*ii*) $\displaystyle E(u_*) = \kappa \int_0^{\mu_*} \left(\frac{1}{y(s)} - \frac{1}{a(s)}\right) ds.$

Corollary 3.4.3. *Let u_1, u_2 be two solutions of the problem* (3.2.3) *corresponding to the solutions $\mu_1 < \mu_2$ of the equation* (3.4.3) *respectively. Then one has*

$$E(u_1) - E(u_2) = -\kappa \int_{\mu_1}^{\mu_2} \left(\frac{1}{y(s)} - \frac{1}{a(s)}\right) ds =: -\frac{1}{p} A_{12} \tag{3.4.5}$$

and

$$A_{12} > 0 \ \Rightarrow \ E(u_1) < E(u_2);$$
$$A_{12} < 0 \ \Rightarrow \ E(u_2) < E(u_1);$$
$$A_{12} = 0 \ \Rightarrow \ E(u_1) = E(u_2).$$

Let us now suppose that we are in case of Figure 3.4.3, then we have:

Theorem 3.4.4. *Let u_1 be the stationary point corresponding to μ_1 such that*

$$a(\mu) < y(\mu) \quad \forall \mu \in (\underline{\mu}, \mu_1), \tag{3.4.6}$$

$$a(\mu) > y(\mu) \quad \forall \mu \in (\mu_1, \overline{\mu}). \tag{3.4.7}$$

Then u_1 is a local minimizer for E. More precisely one has $E(u_1) < E(u)$ $\forall u \neq u_1$, $l(u) \in (\underline{\mu}, \overline{\mu})$.

Proof. By Theorem 3.2.1, we have that

$$\mu_1 = l(u_1), \quad u_1 = \frac{\varphi}{a(\mu_1)^{\frac{1}{p-1}}}. \tag{3.4.8}$$

(*i*) Assume that $l(u) > \mu_1$. Then from (3.4.2), (3.4.7) and Corollary 3.4.2 we have

$$\begin{aligned} E(u) - E(u_1) &= \frac{1}{p}\left(\|\nabla u\|_p^p - \|\nabla u_1\|_p^p\right) - \int_{l(u_1)}^{l(u)} \frac{\kappa}{a(s)} ds \\ &> \frac{1}{p}\left(\|\nabla u\|_p^p - \|\nabla u_1\|_p^p\right) - \int_{l(u_1)}^{l(u)} \frac{\kappa s^{p-1}}{l(\varphi)^{p-1}} ds \\ &= \frac{1}{p}\left(\|\nabla u\|_p^p - \frac{\kappa l(u)^p}{l(\varphi)^{p-1}}\right). \end{aligned} \tag{3.4.9}$$

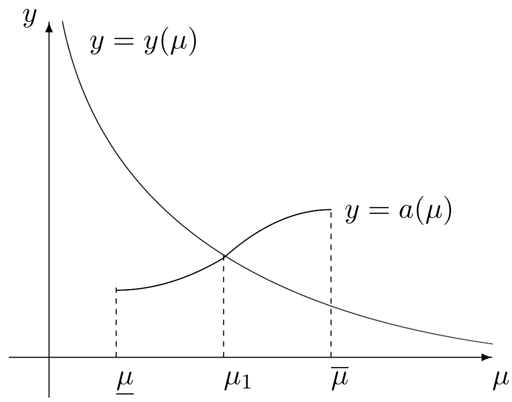


Figure 3.4.3

Note, that Lemma 3.4.1 with $\alpha = 1$ implies that

$$\|\nabla\varphi\|_p = (\kappa l(\varphi))^{\frac{1}{p}}. \tag{3.4.10}$$

Therefore, from assumption (3.4.1), using (3.2.4), Hölder's inequality and (3.4.10), we see

$$\begin{aligned} |l(u)| = \left|\frac{1}{\kappa}\int_\Omega fudx\right| &= \frac{1}{\kappa}\Big|\int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi\nabla udx\Big| \\ &\le \frac{1}{\kappa}\|\nabla\varphi\|_p^{p-1}\|\nabla u\|_p = \frac{l(\varphi)^{\frac{p-1}{p}}}{\kappa^{\frac{1}{p}}}\|\nabla u\|_p. \end{aligned} \tag{3.4.11}$$

Hence, combining (3.4.9), (3.4.11) we derive $E(u) > E(u_1)$ for $l(u) > \mu_1$.

(*ii*) Suppose now $l(u) < \mu_1$. Then as above, we get

$$E(u) - E(u_1) = \frac{1}{p}\left(\|\nabla u\|_p^p - \|\nabla u_1\|_p^p\right) + \int_{l(u)}^{l(u_1)} \frac{\kappa}{a(s)}ds,$$

and by (3.4.6), (3.4.11) and Corollary 3.4.2, we can conclude

$$\begin{aligned} E(u) - E(u_1) &> \frac{1}{p}\left(\|\nabla u\|_p^p - \|\nabla u_1\|_p^p\right) + \int_{l(u)}^{l(u_1)} \frac{\kappa s^{p-1}}{l(\varphi)^{p-1}}ds \\ &= \frac{1}{p}\left(\|\nabla u\|_p^p - \frac{\kappa l(u)^p}{l(\varphi)^{p-1}}\right) - \frac{1}{p}\left(\|\nabla u_1\|_p^p - \frac{\kappa l(u_1)^p}{l(\varphi)^{p-1}}\right) \ge 0. \end{aligned} \tag{3.4.12}$$

Thus, we have $E(u) > E(u_1)$ for $l(u) \in (\underline{\mu}, \overline{\mu})$, $l(u) \neq l(u_1)$. If $l(u) = l(u_1)$ one has by (3.4.1)

$$0 = l(u) - l(u_1) = \int_\Omega g(u - u_1)dx = \frac{1}{\kappa}\int_\Omega f(u - u_1)dx.$$

If this last quantity is vanishing, we will show in Lemma 3.4.6 that $u = u_1$. $\square$

Lemma 3.4.5. *Let u_2 be the stationary point corresponding to μ_2 such that*

$$a(\mu) > y(\mu) \quad \forall \mu \in (\underline{\mu}, \mu_2), \tag{3.4.13}$$

$$a(\mu) < y(\mu) \quad \forall \mu \in (\mu_2, \overline{\mu}) \tag{3.4.14}$$

(see Figure 3.4.4). Then u_2 is a point of local maximum for E in the direction of φ, where φ is the solution of the problem (3.2.4). More precisely one has $E(u_2) > E(u_2 + \delta\varphi)$, for every $\delta \neq 0$ such that

$$\delta \geq -\frac{1}{a(\mu_2)^{\frac{1}{p-1}}} \quad , \quad l(u_2 + \delta\varphi) \in (\underline{\mu}, \overline{\mu}).$$

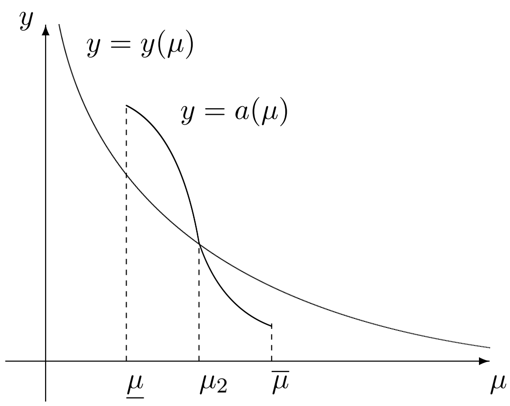


Figure 3.4.4

Proof. As above by Theorem 3.2.1, we have that

$$\mu_2 = l(u_2), \quad u_2 = \frac{\varphi}{a(\mu_2)^{\frac{1}{p-1}}}, \quad u_2 + \delta\varphi = \left(\frac{1}{a(\mu_2)^{\frac{1}{p-1}}} + \delta \right) \varphi. \tag{3.4.15}$$

(i) Let us first assume that $l(u_2 + \delta\varphi) > \mu_2$. Then from (3.4.2), (3.4.14), Lemma

3.4.1 with $\alpha := \dfrac{1}{a(\mu_2)^{\frac{1}{p-1}}} + \delta \geq 0$ and Corollary 3.4.2, we have

$$\begin{aligned} E(u_2+\delta\varphi) - E(u_2) &= \frac{1}{p}\left(\|\nabla(u_2+\delta\varphi)\|_p^p - \|\nabla u_2\|_p^p\right) - \int_{l(u_2)}^{l(u_2+\delta\varphi)} \frac{\kappa}{a(s)} ds \\ &< \frac{1}{p}\left(\|\nabla(u_2+\delta\varphi)\|_p^p - \|\nabla u_2\|_p^p\right) - \int_{l(u_2)}^{l(u_2+\delta\varphi)} \frac{\kappa s^{p-1}}{l(\varphi)^{p-1}} ds \\ &= \frac{1}{p}\left(\|\nabla \alpha\varphi\|_p^p - \frac{\kappa l(\alpha\varphi)^p}{l(\varphi)^{p-1}}\right) - \frac{1}{p}\left(\|\nabla u_2\|_p^p - \frac{\kappa l(u_2)^p}{l(\varphi)^{p-1}}\right) = 0. \end{aligned}$$

Therefore, it holds that

$$E(u_2+\delta\varphi) < E(u_2) \quad \text{for} \quad l(u_2+\delta\varphi) > \mu_2.$$

(ii) Suppose now $l(u_2+\delta\varphi) < \mu_2$. Then similarly, from (3.4.2), (3.4.13), Lemma 3.4.1 and Corollary 3.4.2, we get

$$\begin{aligned} E(u_2+\delta\varphi) - E(u_2) &= \frac{1}{p}\left(\|\nabla(u_2+\delta\varphi)\|_p^p - \|\nabla u_2\|_p^p\right) + \int_{l(u_2+\delta\varphi)}^{l(u_2)} \frac{\kappa}{a(s)} ds \\ &< \frac{1}{p}\left(\|\nabla \alpha\varphi\|_p^p - \frac{\kappa l(\alpha\varphi)^p}{l(\varphi)^{p-1}}\right) - \frac{1}{p}\left(\|\nabla u_2\|_p^p - \frac{\kappa l(u_2)^p}{l(\varphi)^{p-1}}\right) = 0 \end{aligned}$$

as in part *(i)*. Hence,

$$E(u_2+\delta\varphi) < E(u_2) \quad \text{for} \quad l(u_2+\delta\varphi) < \mu_2. \qquad \square$$

Lemma 3.4.6. *Let u be a solution to the problem* (3.2.3)*. Suppose that* (3.1.1) *holds and that $\psi \in W_0^{1,p}(\Omega)$, $\psi \neq 0$ is such that*

$$\int_\Omega f\psi dx = 0. \tag{3.4.16}$$

Then

$$E(u+\psi) > E(u), \tag{3.4.17}$$

i.e., u is a point of minimum for E in any direction of the hyperplane defined by (3.4.16)*.*

Proof. Let us consider ψ which satisfies (3.4.16). Then for $\|\nabla(u+\psi)\|_p > \|\nabla u\|_p$ we have

$$E(u+\psi) - E(u) = \frac{1}{p}\left(\|\nabla(u+\psi)\|_p^p - \|\nabla u\|_p^p\right) > 0.$$

Indeed, from (3.0.3) and (3.4.1), we get

$$\begin{aligned} -\int_{l(u_2)}^{l(u_2+\psi)} \frac{\kappa}{a(s)} ds \geq -\frac{\kappa}{\lambda}(l(u_2+\psi) - l(u_2)) &= -\frac{\kappa}{\lambda}\int_\Omega (g(u_2+\psi) - gu_2) dx \\ &= -\frac{\kappa}{\lambda}\int_\Omega g\psi dx = -\frac{1}{\lambda}\int_\Omega f\psi dx = 0. \end{aligned}$$

Hence, it remains to prove that $\|\nabla(u+\psi)\|_p > \|\nabla u\|_p$. Due to (3.4.16) and since $a > 0$, we get

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \psi dx = 0.$$

Then, proceeding exactly as in the proof of Lemma 2.4.3 we get the result. □

Theorem 3.4.7. *Let* $f \not\equiv 0$, (3.1.1) *holds,* u_2 *be a solution to* (3.2.3) *such that* (3.4.13), (3.4.14) *hold* (*see Figure* 3.4.4, u_2 *corresponds to* μ_2). *Then* u_2 *is a saddle point for the energy* (3.4.2).

Proof. The statement of the theorem is a consequence of Lemmas 3.4.5 and 3.4.6. □

Remark 3.4.1. The same situation occurs if the graph of a is not crossing the graph of y and touching it (see Figure 3.4.5).

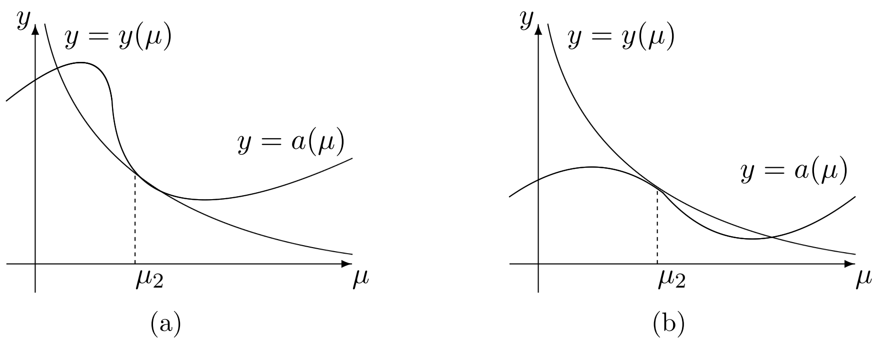


Figure 3.4.5

Lemma 3.4.8. *Let* u *be a weak solution to* (3.0.1) *and suppose that* (3.0.3) *holds. There exists a sequence* t_k *such that* $u_k = u(\cdot, t_k) \to u_\infty$ *in* $W_0^{1,p}(\Omega)$ *as* $t_k \to +\infty$, *where* u_∞ *is a stationary point.*

Proof. Can be proved in the same way as Lemma 2.5.1. □

Corollary 3.4.9. *Suppose that* E *admits a unique global minimizer* u_∞ (u_∞ *is also a solution to the problem* (3.2.3)) *and that the initial value* u_0 *of* (3.0.1) *satisfies* $E(u_0) < E(u_i)$ *for any stationary point* $u_i \neq u_\infty$. *Then* $u(\cdot, t) \to u_\infty$ *in* $W_0^{1,p}(\Omega)$ *as* $t \to +\infty$.

Notice that due to Corollaries 3.4.2, 3.4.3 we can compare the energy at any two different stationary points and we can find a global minimizer of the energy $E(u)$.

3.4.2 Asymptotic behaviour of a strong solution

Theorem 3.4.10. *Let a' be continuous, $2 \le p < +\infty$ and we assume*

$$f = f(x) \in L^2(\Omega),\ u_0 \in W_0^{1,p}(\Omega). \tag{3.4.1}$$

Then for any $T > 0$ there exists a unique strong solution u to (2.0.1) *such that*

$$u \in C([0,T]; W_0^{1,p}(\Omega)), \quad u_t,\ \nabla \cdot |\nabla u|^{p-2}\nabla u \in L^2(0,T;L^2(\Omega)). \tag{3.4.2}$$

Proof. Consider $\varphi_1, \dots, \varphi_n, \dots$ a basis in $W^{2,2(p-1)}(\Omega)$ such that

$$\varphi_i \in H_0^s(\Omega),\ (\varphi_i, v)_{H_0^s(\Omega)} = \mu_i(\varphi_i, v)_{L^2(\Omega)}\ \forall v \in H_0^s(\Omega),$$

where s is chosen in such a way that $H_0^s(\Omega) \subset W_0^{2,2(p-1)}(\Omega)$ (i.e. $\frac{s-2}{n} > \frac{1}{2} - \frac{1}{2(p-1)}$). We assume that $\varphi_1, \dots, \varphi_n, \dots$ is orthonormal in $L^2(\Omega)$. If $u_0 = \sum\limits_i \beta_i \varphi_i$ consider

$$u_n(t) = \sum_{i=1}^n \gamma_i(t)\varphi_i$$

solution to

$$\begin{cases} \displaystyle\int_\Omega u_n' v dx + a(l(u_n)) \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v dx = \int_\Omega f v dx \\ \forall v \in [\varphi_1, \dots, \varphi_n], \\ \displaystyle u_n(0) = \sum_{i=1}^n \beta_i \varphi_i, \end{cases} \tag{3.4.3}$$

where $[\varphi_1, \dots, \varphi_n]$ is the space spanned by $\varphi_1, \dots, \varphi_n$. Taking $v = \varphi_j$ and using the fact that the φ_i's are orthonormal we see that (3.4.3) is equivalent to the Cauchy problem

$$\begin{cases} \displaystyle\gamma_j'(t) = -a\Big(\sum_{i=1}^n \gamma_i(t) \int_\Omega g\varphi_i dx\Big) \int_\Omega \Big|\sum_{i=1}^n \gamma_i(t)\nabla\varphi_i\Big|^{p-2} \sum_{i=1}^n \gamma_i(t)\nabla\varphi_i \nabla\varphi_j dx \\ \displaystyle\qquad\qquad + \int_\Omega f\varphi_j dx, \quad \forall j = 1, \dots n, \\ \gamma_j(0) = \beta_j, \quad \forall j = 1, \dots n. \end{cases} \tag{3.4.4}$$

By the existence theorem for the ordinary differential equations this Cauchy problem possesses a solution $\gamma_j \in C^2([0,\delta))$, $\delta > 0$. Introducing

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_0^{l(u)} \frac{\kappa}{a(s)} ds,$$

we get by (3.4.1) that

$$\frac{d}{dt}E(u(t)) = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla u_t dx - \frac{\kappa}{a(l(u))}\int_\Omega g u_t dx$$
$$= \int_\Omega |\nabla u|^{p-2}\nabla u \nabla u_t dx - \frac{1}{a(l(u))}\int_\Omega f u_t dx$$

Taking $v = u_n'$ in (3.4.3) we get

$$\frac{1}{a(l(u_n))}\int_\Omega |u_n'|^2 dx + \frac{d}{dt}E(u_n(t)) = 0. \tag{3.4.5}$$

By integration we obtain

$$\frac{1}{\lambda}\int_0^t \int_\Omega |u_n'|^2 dxdt \le E(u_n(0)) - E(u_n(t)) \le C, \tag{3.4.6}$$

since E is uniformly bounded from below. Indeed, from (3.4.2) using (3.0.3), Hölder's and Young's inequalities and the fact that $W_0^{1,p}(\Omega) \subset L^2(\Omega)$ for $p \ge 2$ we have that

$$E(u_n) \ge \frac{1}{p}\|\nabla u_n\|_p^p - \frac{\kappa}{\lambda}|g|_2|u_n|_2 \ge \frac{1}{p}\|\nabla u_n\|_p^p - C|g|_2\|\nabla u_n\|_p \ge -\frac{(C|g|_2)^q}{q}$$

Hence, (3.4.6) implies that

$$u_n' \in L^2(0,T;L^2(\Omega)) = L^2(Q_T), \quad Q_T = (0,T)\times\Omega. \tag{3.4.7}$$

We can now differentiate (3.4.3) with respect to t and we get

$$\int_\Omega u_n'' v dx + a'(l(u_n))\int_\Omega g u_n' dx \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v dx$$
$$+ a(l(u_n))\int_\Omega (p-2)|\nabla u_n|^{p-4}\nabla u_n \nabla u_n' \nabla u_n \nabla v + |\nabla u_n|^{p-2}\nabla u_n' \nabla v dx = 0.$$

Taking $v = u_n'$ and noting that the last term is nonnegative we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_n'|^2 dx \le -a'(l(u_n))\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u_n' dx \int_\Omega g u_n' dx. \tag{3.4.8}$$

From the first equation in (3.4.3) written with $v = u_n'$ we have

$$a(l(u_n))\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u_n' dx = \int_\Omega f u_n' dx - \int_\Omega |u_n'|^2 dx$$

and from (3.4.8) follows

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_n'|^2 dx \le \frac{|a'(l(u_n)|}{a(l(u_n))}\left|\left(\int_\Omega f u_n' dx - \int_\Omega |u_n'|^2 dx\right)\int_\Omega g u_n' dx\right|.$$

Since $E(u_n)$ is uniformly bounded so is $l(u_n)$. Due to the fact that $a \in C^1$ from Hölder's inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_n'|^2 dx \leq C\left(\int_\Omega |f|^2 dx + \int_\Omega |u_n'|^2 dx\right)\int_\Omega |u_n'|^2 dx. \tag{3.4.9}$$

Denote by $y_n(t) = |u_n'(t)|_2^2$. Integrating (3.4.9) we get

$$y_n(t) - y_n(s) \leq 2C\int_s^t \big(|f|_2^2 + y_n(\xi)\big)y_n(\xi)d\xi.$$

Passing to the limit in (3.4.6) as $t \to +\infty$ we obtain that

$$\int_0^{+\infty} y_n(s)ds < +\infty.$$

Hence, since $g(x) = 2C(|f|_2^2 x + x^2) > 0$ on $x > 0$ from Lemma A.5 we derive

$$y_n(t) \to 0 \text{ as } t \to +\infty.$$

Thus y_n remains bounded in time. One can complete the proof in the same way as the proof of Theorem 2.5.3. □

3.5 Asymptotic behaviour of some related problems

3.5.1 The Neuman problem

In this section we consider the following problem

$$\begin{cases} u_t - a\left(\int_\Omega gudx\right)\nabla\cdot|\nabla u|^{p-2}\nabla u + \alpha gu = f & \text{in } \Omega\times\mathbb{R}_+, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma\times\mathbb{R}_+, \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{3.5.1}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 1$ with Lipschitz boundary Γ, $\alpha > 0$ and

$$g \geq 0 \text{ in } \Omega,\ g \not\equiv 0. \tag{3.5.2}$$

1. Stationary problem

The stationary problem corresponding to the problem (3.5.1) is the following problem

$$\begin{cases} -a\left(\int_\Omega gudx\right)\nabla\cdot|\nabla u|^{p-2}\nabla u + \alpha gu = f & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma. \end{cases} \tag{3.5.3}$$

Note that integrating on Ω the equation in (3.5.3) we get

$$\int_\Omega gudx = \frac{1}{\alpha}\int_\Omega fdx.$$

Thus there exists a unique stationary point given by the solution to

$$\begin{cases} -a\left(\frac{1}{\alpha}\int_\Omega fdx\right)\nabla\cdot|\nabla u|^{p-2}\nabla u+\alpha gu=f & \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial n}=0 & \text{on } \Gamma.\end{cases} \tag{3.5.4}$$

We denote by φ_a the solution to

$$\begin{cases} -a\nabla\cdot|\nabla \varphi_a|^{p-2}\nabla \varphi_a+\alpha g\varphi_a=f & \text{in } \Omega,\\ |\nabla \varphi_a|^{p-2}\frac{\partial \varphi_a}{\partial n}=0 & \text{on } \Gamma.\end{cases} \tag{3.5.5}$$

The solution of the stationary problem satisfies

$$u=\varphi_{a(l(u))},$$

where

$$l(u):=\int_\Omega gudx=\int_\Omega g\varphi_{a(l(u))}=K(a(l(u))).$$

Integrating (3.5.5) over Ω we see that

$$K(a)=\int_\Omega g\varphi_a dx=\frac{1}{\alpha}\int_\Omega fdx,$$

i.e. K is constant.

2. Positivity of $\Delta_p\varphi_a$

Lemma 3.5.1. *We suppose that* $\frac{f}{g}\in W^{1,p}(\Omega)$ *with*

$$\int_\Omega\left|\nabla\left(\frac{f}{g}\right)\right|^{p-2}\nabla\left(\frac{f}{g}\right)\nabla vdx\ge 0\quad \forall v\in W^{1,p}(\Omega),\ v\ge 0,$$

then it holds that

$$\int_\Omega|\nabla\varphi_a|^{p-2}\nabla\varphi_a\nabla vdx\ge 0\quad \forall v\in W^{1,p}(\Omega),\ v\ge 0.$$

Proof. By definition of φ_a – weak solution – we have

$$a\int_\Omega |\nabla\varphi_a|^{p-2}\nabla\varphi_a\nabla v dx + \alpha\int_\Omega g\varphi_a v dx = \int_\Omega f v dx, \tag{3.5.6}$$

$$a\int_\Omega \left(|\nabla\varphi_a|^{p-2}\nabla\varphi_a - \frac{1}{\alpha^{p-1}}\left|\nabla\left(\frac{f}{g}\right)\right|^{p-2}\nabla\left(\frac{f}{g}\right)\right)\nabla v dx + \alpha\int_\Omega g\left(\varphi_a - \frac{f}{\alpha g}\right)v dx$$
$$= -\frac{a}{\alpha^{p-1}}\int_\Omega \left|\nabla\left(\frac{f}{g}\right)\right|^{p-2}\nabla\left(\frac{f}{g}\right)\nabla v dx$$

(recall that we assumed $g > 0$). Taking in the above inequality

$$v = \left(\varphi_a - \frac{f}{\alpha g}\right)^+$$

by Lemma A.1 we get

$$ac_p\int_\Omega \left(|\nabla\varphi_a| + \left|\nabla\left(\frac{f}{\alpha g}\right)\right|\right)^{p-2}\left|\nabla\left(\varphi_a - \frac{f}{\alpha g}\right)^+\right|^2 dx + \alpha\int_\Omega g\left(\varphi_a - \frac{f}{\alpha g}\right)^{+2} dx \le 0,$$

thus

$$\varphi_a - \frac{f}{\alpha g} \le 0.$$

Coming back to (3.5.6) we obtain

$$a\int_\Omega |\nabla\varphi_a|^{p-2}\nabla\varphi_a\nabla v dx = -\alpha\int_\Omega g\left(\varphi_a - \frac{f}{\alpha g}\right)v dx \ge 0, \quad \forall v \in W^{1,p}(\Omega),\ v \ge 0.$$

This completes the proof of the lemma. □

3. Asymptotic behaviour

Lemma 3.5.2. *Let u be the solution to* (3.5.1). *Assume that a satisfies* (3.1.1), *then it holds that*

$$(u - \varphi_\Lambda)^-,\ (u - \varphi_\lambda)^+ \to 0 \text{ in } L^2(\Omega). \tag{3.5.7}$$

Proof. Denote $\Delta_p u = \nabla\cdot|\nabla u|^{p-2}\nabla u$. From the weak formulations of (3.5.1), (3.5.5) we get

$$u_t - a(l(u(t)))\Delta_p u + \alpha g u = f = -\Lambda\Delta_p\varphi_\Lambda + \alpha g\varphi_\Lambda,$$

hence by Lemma 3.5.1

$$\int_\Omega u_t v dx + a(l(u(t)))\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla\varphi_\Lambda|^{p-2}\nabla\varphi_\Lambda)\nabla v dx + \alpha\int_\Omega g(u - \varphi_\Lambda)v dx$$
$$= \big(\Lambda - a(l(u(t)))\big)\int_\Omega |\nabla\varphi_\Lambda|^{p-2}\nabla\varphi_\Lambda\nabla v dx \ge 0, \quad \forall v \ge 0.$$

Taking $v = -(u-\varphi_\Lambda)^-$ we obtain

$$\frac{1}{2}\frac{d}{dt}|(u-\varphi_\Lambda)^-|_2^2 + \lambda c_p \int_\Omega \big(|\nabla u| + |\nabla \varphi_\Lambda|\big)^{p-2} |\nabla (u-\varphi_\Lambda)^-|^2 dx$$
$$+ \alpha \int_\Omega g((u-\varphi_\Lambda)^-)^2 dx \leq 0,$$

therefore

$$\frac{1}{2}\frac{d}{dt}|(u-\varphi_\Lambda)^-|_2^2 + C|(u-\varphi_\Lambda)^-|_2^2 \leq 0.$$

From above we get

$$|(u-\varphi_\Lambda)^-|_2 \leq |(u_0-\varphi_\Lambda)^-|_2 e^{-Ct}$$

and the result follows. The argument for $(u-\varphi_\lambda)^+$ is the same.

□

Theorem 3.5.3. *Under the assumptions above u the solution to (3.5.1) converges towards the stationary point in* $L^2(\Omega)$*.*

Proof. We have

$$u - \varphi_\Lambda = (u-\varphi_\Lambda)^+ - (u-\varphi_\Lambda)^- \geq -(u-\varphi_\Lambda)^-,$$

$$u - \varphi_\lambda = (u-\varphi_\lambda)^+ - (u-\varphi_\lambda)^- \leq (u-\varphi_\lambda)^+.$$

Combining those two inequalities we get

$$\varphi_\Lambda - (u-\varphi_\Lambda)^- \leq u \leq \varphi_\lambda + (u-\varphi_\lambda)^+.$$

Multiplying by g and integrating over Ω we derive from above

$$l(\varphi_\Lambda) - l((u-\varphi_\Lambda)^-) \leq l(u) \leq l(\varphi_\lambda) + l((u-\varphi_\lambda)^+)$$

and

$$\frac{1}{\alpha}\int_\Omega f dx - l((u-\varphi_\Lambda)^-) \leq l(u) \leq \frac{1}{\alpha}\int_\Omega f dx + l((u-\varphi_\lambda)^+).$$

Using Lemma 3.5.2 we deduce that

$$l(u) \to \frac{1}{\alpha}\int_\Omega f dx$$

as $t \to +\infty$. This implies the convergence result.

□

3.5.2 Problem with average in one dimension

Let us consider the following problem

$$\begin{cases} u_t - \nabla \cdot \left(a \left(\fint_{\Omega(x,r)} u dx \right) \nabla u \right) = f & \text{in } \Omega \times \mathbb{R}_+, \\ u = 0 & \text{on } \Gamma \times \mathbb{R}_+, \\ u(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases} \tag{3.5.8}$$

where $\Omega \subset \mathbb{R}$ is a bounded domain with the smooth boundary Γ, $\Omega(x,r) = \Omega \cap B(x,r)$. $B(x,r)$ denotes a ball with the center at the point x and the radius r.

The corresponding to (3.5.8) stationary problem is the problem of finding a solution to

$$\begin{cases} -\nabla \cdot \left(a \left(\fint_{\Omega(x,r)} u dx \right) \nabla u \right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{3.5.9}$$

We consider $0 < r < d$, where d is a diameter of Ω. We assume that

$$0 < \lambda \leq a(s) \tag{3.5.10}$$

and there exists $L > 0$ such that

$$|a(s) - a(t)| \leq L|s - t|, \quad \forall s, t \in \mathbb{R}. \tag{3.5.11}$$

Assume there exists a unique solution $u_\infty \in H_0^1(\Omega)$ to the stationary problem (existence and uniqueness results can be found in [2], [15]), then the following theorem holds.

Theorem 3.5.4. *Let $\Omega \subset \mathbb{R}$ be a bounded domain, $f = f(x) \in H^{-1}(\Omega)$, $u_0 \in L^2(\Omega)$ and $L < \frac{\lambda^2}{\|f\|_* d^{\frac{1}{2}}}$. Then it holds that $\forall r > 0$*

$$u(t) \to u_\infty \text{ in } L^2(\Omega) \text{ when } t \to +\infty.$$

Proof. From (3.5.8), (3.5.9) we derive

$$\int_\Omega u_t v dx + \int_\Omega a \left(\fint_{\Omega(x,r)} u(y,t) dy \right) \nabla u \nabla v dx$$
$$= \int_\Omega a \left(\fint_{\Omega(x,r)} u_\infty(y,t) dy \right) \nabla u_\infty \nabla v dx, \quad \forall v \in H_0^1(\Omega).$$

Since u_∞ is independent of t and taking $v = u - u_\infty$ we have

$$\int_\Omega (u-u_\infty)_t(u-u_\infty)dx + \int_\Omega a\left(\fint_{\Omega(x,r)} u(y,t)dy\right)|\nabla(u-u_\infty)|^2dx$$
$$= \int_\Omega \left(a\left(\fint_{\Omega(x,r)} u_\infty(y,t)dy\right) - a\left(\fint_{\Omega(x,r)} u(y,t)dy\right)\right)\nabla u_\infty \nabla(u-u_\infty)dx.$$

Thus

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u-u_\infty|^2dx + \int_\Omega a\left(\fint_{\Omega(x,r)} u(y,t)dy\right)|\nabla(u-u_\infty)|^2dx =$$
$$= \int_\Omega \left|a\left(\fint_{\Omega(x,r)} u_\infty(y,t)dy\right) - a\left(\fint_{\Omega(x,r)} u(y,t)dy\right)\right| |\nabla u_\infty||\nabla(u-u_\infty)|dx.$$

Recalling (3.5.10), (3.5.11) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u-u_\infty|^2dx + \lambda\int_\Omega |\nabla(u-u_\infty)|^2dx$$
$$\leq L\int_\Omega \left(\fint_{\Omega(x,r)} |u_\infty - u|(y,t)dy\right)|\nabla u_\infty||\nabla(u-u_\infty)|dx. \quad (3.5.12)$$

From Morrey's inequality and since $(u-u_\infty) \in H_0^1(\Omega)$ we get

$$\fint_{\Omega(x,r)} |u_\infty - u|(y,t)dy \leq \fint_{\Omega(x,r)} \sup_\Omega |u_\infty - u|(y,t)dy$$
$$\leq \fint_{\Omega(x,r)} d^{\frac{1}{2}}\left(\int_\Omega |\nabla(u_\infty - u)|^2dx\right)^{\frac{1}{2}}dy = d^{\frac{1}{2}}\left(\int_\Omega |\nabla(u_\infty - u)|^2dx\right)^{\frac{1}{2}}.$$

Going back to (3.5.12) and using Cauchy-Schwarz inequality it comes

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u-u_\infty|^2dx + \lambda\int_\Omega |\nabla(u-u_\infty)|^2dx$$
$$\leq Ld^{\frac{1}{2}}\int_\Omega |\nabla(u_\infty - u)|^2dx\left(\int_\Omega |\nabla u_\infty|^2dx\right)^{\frac{1}{2}}. \quad (3.5.13)$$

To estimate $\left(\int_\Omega |\nabla u_\infty|^2dx\right)^{\frac{1}{2}}$ recall that u_∞ satisfies

$$\int_\Omega a\left(\fint_{\Omega(x,r)} u_\infty(y,t)dy\right)\nabla u_\infty \nabla v dx = \int_\Omega fvdx \quad \forall v \in H_0^1(\Omega).$$

Taking $v = u_\infty$ we derive

$$\lambda\|\nabla u_\infty\|_2^2 \leq \|f\|_*\|\nabla u_\infty\|_2,$$

where $\|\cdot\|_*$ denotes a dual norm. Thus

$$\|\nabla u_\infty\|_2 \leq \frac{\|f\|_*}{\lambda}. \tag{3.5.14}$$

From (3.5.13), (3.5.14) we obtain

$$\frac{1}{2}\frac{d}{dt}\|u-u_\infty\|_2^2 + \lambda\|\nabla(u-u_\infty)\|_2^2 \leq Ld^{\frac{1}{2}}\frac{\|f\|_*}{\lambda}\|\nabla(u-u_\infty)\|_2^2$$

and

$$\frac{1}{2}\frac{d}{dt}\|u-u_\infty\|_2^2 + \left(\lambda - Ld^{\frac{1}{2}}\frac{\|f\|_*}{\lambda}\right)\|\nabla(u-u_\infty)\|_2^2 \leq 0.$$

Since $L < \dfrac{\lambda^2}{\|f\|_* d^{\frac{1}{2}}}$, then $\alpha = \lambda - Ld^{\frac{1}{2}}\dfrac{\|f\|_*}{\lambda} > 0$. Using Poincaré's inequality we get

$$\frac{1}{2}\frac{d}{dt}\|u-u_\infty\|_2^2 + \frac{\alpha}{C_p}\|u-u_\infty\|_2^2 \leq 0,$$

where $C_p > 0$ is the Poincaré constant, which depends only on domain Ω. This implies

$$\frac{d}{dt}\|u-u_\infty\|_2^2 + \frac{2\alpha}{C_p}\|u-u_\infty\|_2^2 \leq 0,$$

and

$$\|u-u_\infty\|_2^2 \leq e^{-\frac{2\alpha t}{C_p}}\|u_0-u_\infty\|_2^2.$$

Thus

$$\|u-u_\infty\|_2 \leq e^{-\frac{\alpha t}{C_p}}\|u_0-u_\infty\|_2,$$

which completes the proof. $\square$

Chapter 4

Local versus nonlocal interactions in a reaction-diffusion system of population dynamics

In this chapter we address the following reaction-diffusion system with nonlocal interaction (see [30, 31, 43]):

$$\begin{cases} u_t = u_{xx} + u(1-u) - uv & \text{in } \Omega \times \mathbb{R}_+ \\ v_t = \lambda v_{xx} - \chi (u_x v)_x - \beta v + \delta \, \dfrac{\langle u, v \rangle}{\langle 1, v \rangle} \, v - \gamma \, \dfrac{uv}{1 + \tau v} & \text{in } \Omega \times \mathbb{R}_+ \\ u_x = v_x = 0 & \text{in } \partial\Omega \times \mathbb{R}_+ \\ u = u_0, \ v = v_0 & \text{in } \Omega \times \{0\} \, . \end{cases} \tag{4.0.1}$$

Here $\Omega \equiv (0,1)$, $\mathbb{R}_+ \equiv (0,\infty)$, $\partial\Omega \equiv \{0,1\}$, β, γ, δ, τ are positive constant coefficients, $\lambda > 0$ and $\chi \geq 0$ will be regarded as parameters, and

$$\langle u, v \rangle (t) := \int_0^1 u(x,t) v(x,t) \, dx \, , \quad \langle 1, v \rangle (t) := \int_0^1 v(x,t) \, dx \qquad (t \in \mathbb{R}_+) \tag{4.0.2}$$

for any measurable $u, v : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$. The unknowns $v = v(x,t)$ and $u = u(x,t)$ denote the densities of a population of amoebae, feeding on bacteria, respectively of bacteria belonging to a virulent strain, which can kill amoebae by infecting them - a novel feature with respect to standard predator-prey interaction (in fact, amoebae are attacked by bacteria following a Holling type II functional response, with handling time τ and attack rate γ). However, the main feature of the model is that predation of the amoeboid population on bacteria is governed by a nonlocal law through the integral term $\delta \dfrac{\langle u,v\rangle}{\langle 1,v\rangle} v$. This describes the fact that amoebae behave like a sole organism when food supply is low, so that food is redistributed among all cells (see [30] for a discussion of this point).

The question we want to address in this chapter is that of existence of *patterns*, namely, of space dependent stable equilibrium solutions of problem (4.0.1). Both experimental and numerical evidence support the existence of such solutions, which is related to the pathogenic action of bacteria [31]. Specifically, we wonder whether existence or nonexistence of patterns is affected by the nonlocal character of the interaction. Therefore, we also investigate existence of patterns for the companion problem

$$\begin{cases} u_t = u_{xx} + u(1-u) - uv & \text{in } \Omega \times \mathbb{R}_+ \\ v_t = \lambda v_{xx} - \chi(u_x v)_x - \beta v + \delta uv - \gamma \dfrac{uv}{1+\tau v} & \text{in } \Omega \times \mathbb{R}_+ \\ u_x = v_x = 0 & \text{in } \partial\Omega \times \mathbb{R}_+ \\ u = u_0,\ v = v_0 & \text{in } \Omega \times \{0\}\,, \end{cases} \tag{4.0.3}$$

where the *nonlocal term* $\delta \dfrac{\langle u,v\rangle}{\langle 1,v\rangle} v$ is replaced by the *local interaction* term δuv. Our approach is to treat both (4.0.1) and (4.0.3) as bifurcation problems, the bifurcation parameter being the *diffusivity* λ of amoebae, and investigate the possibility of patterns bifurcating out of a constant steady state solution. As we shall see, a second relevant parameter in this analysis is the strength χ of the *chemotactic term* $-\chi(u_x v)_x$.

Both systems (4.0.1) and (4.0.3) are space dependent generalizations of the "lumped parameter" Cauchy problem

$$\begin{cases} \dot{u} = u(1-u) - uv & \text{in } \mathbb{R}_+ \\ \dot{v} = -\beta v + \delta uv - \gamma \dfrac{uv}{1+\tau v} & \text{in } \mathbb{R}_+ \\ u(0) = u_0,\ v(0) = v_0\,. \end{cases} \tag{4.0.4}$$

Steady state solutions of problem (4.0.4) are spatially homogeneous equilibria of both problems (4.0.1) and (4.0.3). In particular, we shall be interested in *coexistence equilibria* of (4.0.4) - namely, in steady state solutions $\bar{U} \equiv (\bar{u}, \bar{v})$ such that $\bar{u}, \bar{v} > 0$. In the following we assume that there exists a coexistence steady state $\bar{U}$, with $0 < \bar{u} < 1,\ 0 < \bar{v} < 1$, which is asymptotically stable with respect to problem (4.0.4) (note that we can do it, since this occurs for a suitable choice of parameters β, δ, γ, τ (see [30])). Then we seek conditions on the parameters λ and χ ensuring that:

(i) the steady state $\bar{U}$ becomes unstable with respect to solutions of problem (4.0.3) - namely, *Turing destabilization* of $\bar{U}$ occurs;

(ii) patterns of problem (4.0.3) bifurcate from $\bar{U}$.

Subsequently, the same question is addressed for problem (4.0.1), to study whether the conditions of Turing destabilization are affected by the nonlocal character of the interaction.

The main qualitative outcome of the above analysis is that the nonlocal interaction enhances the possibility that patterns exist with respect to the case of local interaction. In fact, in the local case patterns can only exist for small values of the parameter χ (see assumption (A_1)), a requirement which has no counterpart in the nonlocal case. Moreover, in the local case patterns can exist for a more limited range of values of the diffusivity λ than in the nonlocal case (this is apparent from the subsequent discussion, since the function $\tilde{\psi}$ defined in (4.1.20) is always smaller than the function ψ defined in (4.1.9)). It is worth observing that these results are in agreement with those proven in [32] for a single reaction-diffusion equation with nonlocal interaction, showing that for such an equation patterns can exist in cases where this is impossible, if a local interaction is considered [10, 40].

4.1 Results

4.1.1 Well-posedness

Let $C^k(\overline{\Omega})$ denote the space of k times continuously differentiable functions $u : \overline{\Omega} \to \mathbb{R}$, endowed with the usual norm ($k \in \mathbb{N} \cup \{0\}$; $C(\overline{\Omega}) \equiv C^0(\overline{\Omega})$).

Solutions of problems (4.0.1) and (4.0.3) are always meant in the classical sense. The following well-posedness result for problem (4.0.1) is easily proven. A companion result holds for problem (4.0.3), whose formulation is left to the reader.

Theorem 4.1.1. *For any $u_0, v_0 \in C(\overline{\Omega})$, $u_0 \geq 0$, $v_0 \geq 0$ there exists a unique global solution (u, v) of problem* (4.0.1)*. Moreover, there holds $u > 0$, $v > 0$ in $\Omega \times \mathbb{R}_+$.*

4.1.2 Existence of patterns: local interaction

Let us first address the simpler problem (4.0.3).

Steady state solutions of problem (4.0.4) are found solving the system

$$\begin{cases} F(u,v) := u(1-u-v) = 0 \\ G(u,v) := \left(-\gamma \dfrac{u}{1+\tau v} + \delta u - \beta\right) v = 0\,. \end{cases} \tag{4.1.1}$$

In particular, coexistence equilibria of problem (4.0.4) are found solving the system

$$\begin{cases} u = 1 - v \\ \gamma \dfrac{u}{1+\tau v} - \delta u + \beta = 0\,. \end{cases} \tag{4.1.2}$$

Hereafter we set $F_u \equiv F_u(\bar{U}), F_v \equiv F_v(\bar{U}), G_u \equiv G_u(\bar{U}), G_v \equiv G_v(\bar{U})$. There holds

$$F_u = F_v = -\bar{u}\,, \tag{4.1.3}$$

$$G_u = \left(\delta - \frac{\gamma}{1+\tau\bar{v}}\right)\bar{v}\,, \tag{4.1.4}$$

$$G_v = \frac{\gamma\tau}{(1+\tau\bar{v})^2}\,\bar{u}\bar{v}\,. \tag{4.1.5}$$

Denote by

$$J \equiv J(\bar{u},\bar{v}) := \begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix} \tag{4.1.6}$$

the linearized operator of the right-hand side on the solution $\bar{U}$. By standard results, $\bar{U}$ is asymptotically stable with respect to the ODE problem (4.0.4) if

$$(A_0) \qquad F_u + G_v < 0\,, \quad G_u > G_v\,;$$

in fact, the above conditions ensure that

$$Tr\,J = F_u + G_v < 0\,, \quad Det\,J = F_uG_v - F_vG_u > 0\,.$$

Let $\bar{U}$ be a solution of (4.1.2). We wonder whether the Turing destabilization of $\bar{U}$, regarded as a spatially homogeneous equilibrium of (4.0.3), occurs for some values of the parameters λ and χ. It turns out that this can only happen if

$$(A_1) \qquad 0 \le \chi < \chi_0 := \frac{G_v}{\bar{v}|F_v|} = \frac{\gamma\tau}{(1+\tau\bar{v})^2}\,,$$

and

$$(A_2) \qquad 0 < \lambda < \lambda_0 := \frac{1}{|F_u|}\left(2G_u - G_v + \chi\bar{v}F_v - 2\sqrt{(G_u + \chi\bar{v}F_v)(G_u - G_v)}\right)$$

(observe that λ_0 is well defined and positive by assumptions (A_0)-(A_1)). More precisely, we have the following result.

Theorem 4.1.2. *Let $\bar{U} \equiv (\bar{u},\bar{v})$ be a stationary solution of problem* (4.0.4) *such that $0 < \bar{u} < 1,\ 0 < \bar{v} < 1$, and let assumption (A_0) be satisfied. Then the homogeneous steady state $\bar{U}$ is unstable with respect to problem* (4.0.3) *if and only if:*
(i) the chemotaxis coefficient χ satisfies condition (A_1), and the diffusion coefficient λ of amoebae satisfies condition (A_2);
(ii) there exists $n \in \mathbb{N}$ such that

$$k_-(\lambda,\chi) < k_n := n^2\pi^2 < k_+(\lambda,\chi)\,, \tag{4.1.7}$$

where

$$k_\pm(\lambda,\chi) := \frac{1}{2\lambda}\left\{F_u\lambda + G_v + \chi\bar{v}F_v \pm \sqrt{(F_u\lambda + G_v + \chi\bar{v}F_v)^2 + 4\lambda F_u(G_u - G_v)}\right\}\,. \tag{4.1.8}$$

Observe that the functions $k_\pm$ defined in (4.1.7) are the roots of the equation

$$\psi(\lambda, \chi, k) := \lambda k^2 - (F_u\lambda + G_v + \chi\bar{v}F_v)k - F_u(G_u - G_v) = 0\,. \tag{4.1.9}$$

By assumption (A_0) and equality (4.1.3) there holds

$$\psi(\lambda, \chi, 0) = F_u(G_v - G_u) = |F_u|(G_u - G_v) > 0\,, \tag{4.1.10}$$

thus positive roots of equation (4.1.9) need not exist. Existence prevails, if assumptions (A_1)-(A_2) are satisfied; in fact, in this case there holds $0 < k_-(\lambda, \chi) < k_+(\lambda, \chi)$ (see Section 4.2).

In the following of this subsection we assume $\chi \in [0, \chi_0)$ to be fixed. Accordingly, for any fixed $\chi \in [0, \chi_0)$ we set $\psi(\lambda, k) \equiv \psi(\lambda, \chi, k)$ and $k_\pm(\lambda) \equiv k_\pm(\lambda, \chi)$. An elementary analysis shows that (see Figure 4.1.1):

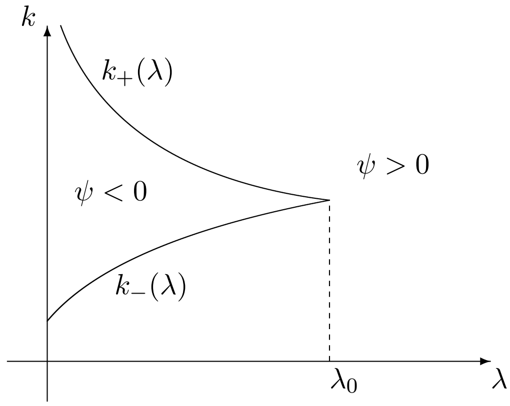


Figure 4.1.1

(a) k_- is increasing, k_+ decreasing with $\lambda \in (0, \lambda_0)$ and

$$k_\pm(\lambda_0) = \frac{F_u\lambda_0 + G_v + \chi\bar{v}F_v}{2\lambda_0}\,; \tag{4.1.11}$$

(b) there holds

$$\lim_{\lambda\to 0^+} k_-(\lambda) = \frac{|F_u|(G_u - G_v)}{G_v + \chi\bar{v}F_v} > 0\,, \qquad \lim_{\lambda\to 0^+} k_+(\lambda) = \infty\,. \tag{4.1.12}$$

The proof of Theorem 4.1.2 relies on a linearized stability analysis of problem (4.0.3). The Fréchet derivative of the system in (4.0.3) at $\bar{U} \equiv (\bar{u}, \bar{v})$ is the operator-valued matrix

$$\begin{pmatrix} \dfrac{d^2}{dx^2} + F_u & F_v \\ -\chi\bar{v}\dfrac{d^2}{dx^2} + G_u & \lambda\dfrac{d^2}{dx^2} + G_v \end{pmatrix}$$

(see (4.2.4)), supplemented with homogeneous Neumann boundary conditions. Its spectrum consists of eigenvalues $\zeta_n \in \mathbb{C}$, which are the roots of the equation

$$\zeta^2 + \phi(\lambda, k_n)\zeta + \psi(\lambda, k_n) = 0 \tag{4.1.13}$$

where $k_n := n^2\pi^2$ ($n \in \mathbb{N} \cup \{0\}$) and

$$\phi(\lambda, k) := (1+\lambda)k - (F_u + G_v)\,. \tag{4.1.14}$$

Clearly, we are interested in eigenvalues ζ_n with positive real part, which turn out to be real positive solutions of equation (4.1.13). The proof of Theorem 4.1.2 shows that such solutions exist if and only if the conditions (i)-(ii) of the theorem are satisfied.

Now suppose that assumptions (A_0)-(A_2) are satisfied. By (4.1.12) inequality (4.1.7) is satisfied for any $n \in \mathbb{N}$ sufficiently large, thus by Theorem 4.1.2 $\bar{U}$ is unstable for any $\lambda > 0$ sufficiently small. Then it is natural to conjecture that a pattern of problem (4.0.3) bifurcates from $\bar{U}$ at some value $\lambda \in (0, \lambda_0]$.

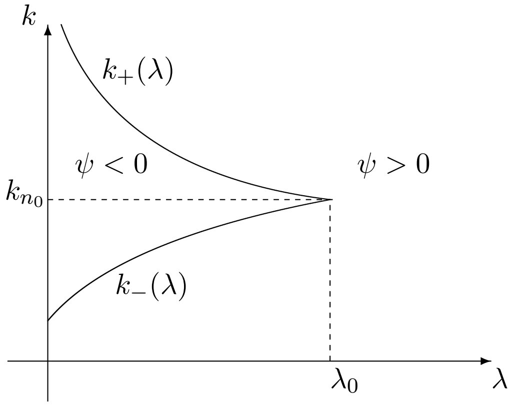


Figure 4.1.2: Condition (4.1.15)

The above question can be addressed by standard methods of bifurcation theory ($e.g.$, see [1, 36]). In fact, let there exist $n_0 \in \mathbb{N}$ and $\tilde{\lambda}_0 \in (0, \lambda_0]$ such that $k_{n_0} = k_-(\tilde{\lambda}_0)$. Then $\psi(\tilde{\lambda}_0, k_{n_0}) = \psi(\tilde{\lambda}_0, k_-(\tilde{\lambda}_0)) = 0$, and $\zeta(k_{n_0}) = 0$ is an eigenvalue of the operator $A_{\tilde{\lambda}_0}$. To avoid technicalities, we only consider the case when this eigenvalue is simple. This is certainly the case if $\tilde{\lambda}_0 = \lambda_0$, since for any $n \in \mathbb{N} \setminus \{n_0\}$ there holds $\psi(\lambda_0, k_n) > 0$, thus the real part of $\zeta(k_n)$ is negative (see Figure 4.1.2). Then we have the following result (see Figure 4.1.3), where the labels s and u stand for "stable" and "unstable", respectively, and E denotes the eigenvector (4.2.9)).

Theorem 4.1.3. *Let $\bar{U}$ be the homogeneous steady state considered in Theorem 4.1.2. Let assumptions (A_0)-(A_2) be satisfied. Moreover, suppose that*

$$\text{there exists } n_0 \in \mathbb{N} \text{ such that } k_{n_0} = k_\pm(\lambda_0)\,. \tag{4.1.15}$$

Then:

(i) $(\lambda_0, \bar{U})$ *is a bifurcation point of stationary solutions of problem* (4.0.3) *;*

(ii) the bifurcating stationary solutions are nonconstant, and exist in some neighbourhood of the bifurcation point $(\lambda_0, \bar{U})$*;*

(iii) the bifurcation is subcritical, and the bifurcating nonconstant stationary solutions are asymptotically stable.

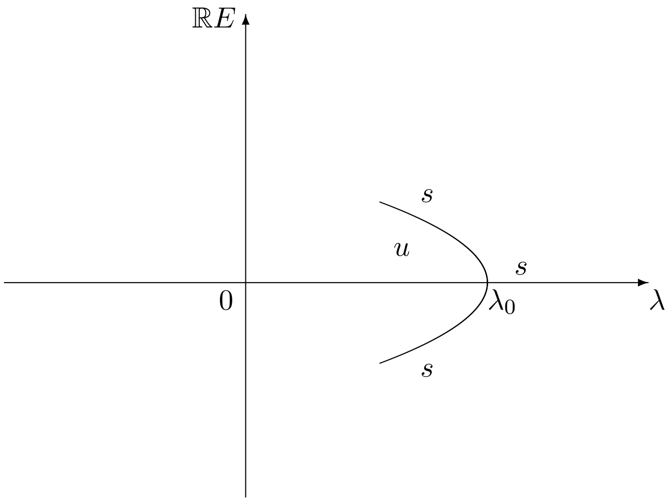


Figure 4.1.3: Theorem 4.1.3

Remark 4.1.1. Concerning statement *(ii)* of Theorem 4.1.3, the set of bifurcating solutions can be described as follows (see [1, Proposition 26.13]). Denote by Y the Banach space

$$Y := \{U \equiv (u, v) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega}) | u'(0) = v'(0) = u'(1) = v'(1) = 0\} \tag{4.1.16}$$

with norm

$$\|U\|_Y := \sum_{k=0}^{2} \left\{ \|u^{(k)}\|_\infty + \|v^{(k)}\|_\infty \right\}$$

for any $U \equiv (u, v) \in Y$. Then there exist $\varepsilon > 0$ and a smooth map $U : (-\varepsilon, \varepsilon) \to Y$ such that for any $s \in (-\varepsilon, \varepsilon)$ and $x \in \overline{\Omega}$ the bifurcating stationary solutions are given by the equality

$$U(s, x) = \bar{U} \; + \; s \left[\left(\cos\left(\sqrt{k_{n_0}}\, x\right), \frac{k_{n_0} - F_u}{F_v} \; \cos\left(\sqrt{k_{n_0}}\, x\right) \right) + y(s, x) \right] , \tag{4.1.17}$$

where the map $s \to y(s,\cdot)$ belongs to $C^1\big((-\varepsilon,\varepsilon);N_c\big)$ for some closed subspace $N_c \subseteq Y$, and $y(0,\cdot)=0$. Moreover, there exists a smooth map $\lambda : (-\varepsilon,\varepsilon) \to \mathbb{R}_+$ such that $\lambda=\lambda(s)$ for any $s \in (-\varepsilon,\varepsilon)$, $\lambda \in \mathbb{R}_+$ being the parameter in problem (4.0.3) and $\lambda_0=\lambda(0)$.

In view of Theorem 4.1.3 and Remark 4.1.1, there exists $\varepsilon>0$ such that for any $\lambda \in (\lambda_0-\varepsilon,\lambda_0)$ there exist patterns of problem (4.0.3). Observe that, under the assumptions of Theorem 4.1.3, the steady state $\bar{U}$ is unstable with respect to problem (4.0.3) for any $\lambda \in (0,\lambda_0)$, whereas it is asymptotically stable for any $\lambda>\lambda_0$ (see Theorem 4.1.2 and Figure 4.1.3).

Remark 4.1.2. Conclusions similar to those of Theorem 4.1.3 hold in more general situations. In fact, let there exist $n_0 \in \mathbb{N}$ such that

$(A_3)(i)$
$$k_-(0) < k_{n_0} < k_\pm(\lambda_0)\,.$$

Then there exists a unique $\tilde{\lambda}_0 \in (0,\lambda_0)$ such that $k_{n_0}=k_-(\tilde{\lambda}_0)$ (see Figure 4.1.4a). Suppose that

$(A_3)(ii)$
$$k_{n_0+1} > k_+(\tilde{\lambda}_0)\,.$$

Since the function ψ is increasing in $(k_\pm(\lambda_0),\infty)$, this implies that $\psi(k_n)>0$ for every $n \geq n_0+1$. Plainly, it follows that the real part of $\zeta(k_n)$ is negative for any $n \in \mathbb{N}\setminus\{n_0\}$. Then the same conclusions of Theorem 4.1.3 hold with λ_0 replaced by $\tilde{\lambda}_0$. Similar remarks hold in analogous situations (*e.g.*, see Figure 4.1.4b); we leave their formulation to the reader.

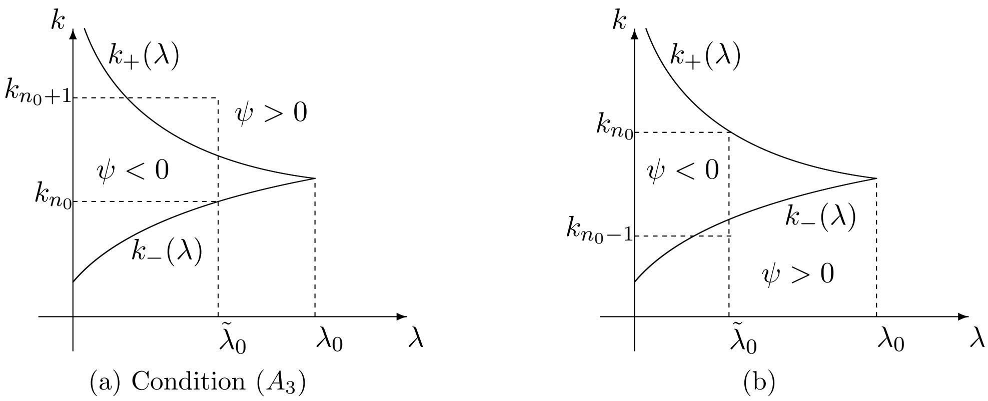


Figure 4.1.4

4.1.3 Existence of patterns: nonlocal interaction

Let us now regard the coexistence steady state $\bar{U}$ as a spatially homogeneous equilibrium of problem (4.0.1). It will be seen below that in this case the functions $k_\pm$

of the previous analysis (see (4.1.8)) are replaced by

$$\tilde{k}_{\pm}(\lambda,\chi) := \frac{1}{2\lambda}\left\{F_u\lambda + G_v + \chi\bar{v}F_v \pm \sqrt{(F_u\lambda + G_v + \chi\bar{v}F_v)^2 - 4\lambda F_u(G_v - G_u + \delta\bar{v})}\right\}, \tag{4.1.18}$$

which are the roots of the equation

$$\tilde{\psi}(\lambda,\chi,k) = 0\,; \tag{4.1.19}$$

here

$$\begin{aligned}\tilde{\psi}(\lambda,\chi,k) &:= \psi(\lambda,\chi,k) + \delta\bar{v}F_v = \\ &= \lambda k^2 - (F_u\lambda + G_v + \chi\bar{v}F_v)k + F_u(G_v - G_u + \delta\bar{v})\end{aligned} \tag{4.1.20}$$

Observe that, at variance from the previous case (see (4.1.10)), there holds

$$\tilde{\psi}(\lambda,\chi,0) = F_u(G_v - G_u + \delta\bar{v}) = -\gamma\,\frac{\bar{u}\bar{v}}{(1+\tau\bar{v})^2}\,[1+\tau(\bar{u}+\bar{v})] < 0\,. \tag{4.1.21}$$

Hence for any $\lambda > 0$ and $\chi \geq 0$

$$\tilde{k}_-(\lambda,\chi) < 0 < \tilde{k}(\lambda,\chi) \tag{4.1.22}$$

(see Figure 4.1.5, where $\chi \geq 0$ is fixed). The root $\tilde{k}_-$ has no role in the subsequent analysis since it is always negative, thus we set $\tilde{k} \equiv \tilde{k}_+$ hereafter. Observe that assumptions (A_1)-(A_2) have no counterpart in the present case. However, it is worth mentioning that

$$\lim_{\lambda\to 0^+}\tilde{k}_-(\lambda) = \begin{cases} \dfrac{F_u(G_v - G_u + \delta\bar{v})}{G_v + \chi\bar{v}F_v} < 0 & \text{if } (A_1) \text{ holds} \\ -\infty & \text{otherwise}\,. \end{cases}$$

Let $\chi \geq 0$ be fixed, and set $\tilde{\psi}(\lambda,k) \equiv \tilde{\psi}(\lambda,\chi,k)$, $\tilde{k}(\lambda) \equiv \tilde{k}(\lambda,\chi)$. It is easily checked that $\tilde{k}$ is decreasing in $(0,\infty)$, and

$$\lim_{\lambda\to 0^+}\tilde{k}(\lambda) = \infty\,, \qquad \lim_{\lambda\to\infty}\tilde{k}(\lambda) = 0\,.$$

Denote by $\lambda_1 \in (0,\infty)$ the unique root of the equation $\tilde{k}(\lambda) = k_1$, namely

$$\lambda = \lambda_1 \iff \tilde{k}(\lambda) = k_1 \tag{4.1.23}$$

(recall that $k_1 := \pi^2$). Arguing as in Subsection 4.1.2, we obtain the following result.

Theorem 4.1.4. *Let $\bar{U} \equiv (\bar{u},\bar{v})$ be a stationary solution of problem* (4.0.4) *such that $0 < \bar{u} < 1$, $0 < \bar{v} < 1$, and let assumption (A_0) be satisfied. Let $\lambda_1 \in (0,\infty)$ be the unique root of the equation $\tilde{k}(\lambda) = k_1$. Then the homogeneous steady state $\bar{U}$ is unstable with respect to problem* (4.0.1) *if and only if $\lambda \in (0,\lambda_1)$.*

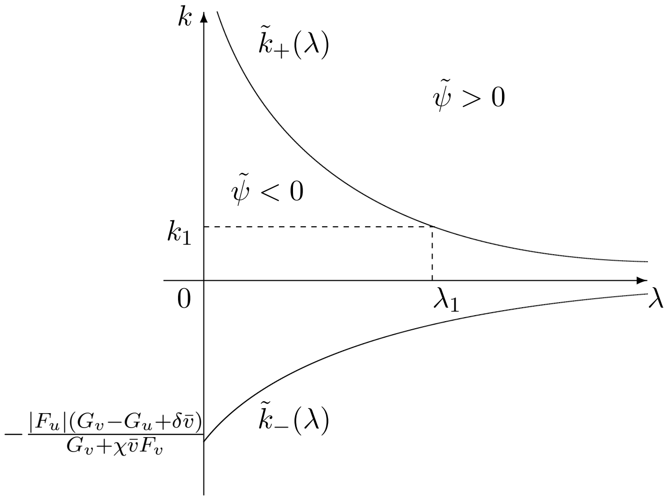


Figure 4.1.5

As in the case of local interaction, the proof of Theorem 4.1.4 is based on a linearized stability analysis of problem (4.0.1). The Fréchet derivative of the system in (4.0.1) at $\bar{U} \equiv (\bar{u}, \bar{v})$ is the operator-valued matrix

$$\begin{pmatrix} \dfrac{d^2}{dx^2} + F_u & F_v \\ & \\ -\chi\bar{v}\dfrac{d^2}{dx^2} + G_u + \delta\bar{v}\left[\langle 1, \cdot\rangle - 1\right] & \lambda\dfrac{d^2}{dx^2} + G_v \end{pmatrix} \tag{4.1.24}$$

(see Section 4.3), supplemented with homogeneous Neumann boundary conditions; here the linear functional $\langle 1, \cdot\rangle$ is defined in (4.0.2), $F_u \equiv F_u(\bar{u}, \bar{v})$, and so on. By analogy with the situation encountered for the case of local interaction, it is natural to conjecture that $(\bar{U}, \lambda_1)$ be a bifurcation point of patterns of problem (4.0.1). The affirmative answer is the content of the following theorem.

Theorem 4.1.5. *Let $\bar{U}$ be the homogeneous steady state considered in Theorem 4.1.2, and let assumption (A_0) be satisfied. Let $\lambda_1 \in (0, \infty)$ be the unique root of the equation $\tilde{k}(\lambda) = k_1$. Then the conclusions of Theorem 4.1.3 hold true, with λ_0 replaced by λ_1. Moreover, the nonconstant bifurcating stationary solutions are of the form* (4.1.17) *with k_{n_0} replaced by k_1.*

4.2 Local interaction: proofs

Consider the Banach space $X := C(\overline{\Omega}) \times C(\overline{\Omega})$ endowed with the norm

$$\|U\|_X := \|u\|_\infty + \|v\|_\infty \qquad (U \equiv (u,v) \in X)\,.$$

Define a bounded nonlinear operator $\mathcal{F} : \mathbb{R}_+ \times \overline{\mathbb{R}}_+ \times Y \to X$, with Y as in (4.1.16)), by setting

$$\mathcal{F}(\lambda,\chi,U) \;:=\; \begin{pmatrix} u'' + F(u,v) \\ \lambda v'' - \chi(u'v)' + G(u,v) \end{pmatrix} \tag{4.2.1}$$

for any $\lambda > 0$, $\chi \geq 0$ and $U \equiv (u,v) \in Y$. Then problem (4.0.3) reads as the abstract Cauchy problem

$$\begin{cases} U_t = \mathcal{F}(\lambda,\chi,U) \quad \text{in } \mathbb{R}_+ \\ U(0) \;= U_0 := (u_0,v_0)\,. \end{cases} \tag{4.2.2}$$

Proof of Theorem 4.1.1. For every $\lambda > 0$, $\chi \geq 0$ the map $\mathcal{F}(\lambda,\chi,\cdot) : Y \to X$ is locally Lipschitz continuous, thus for each $U_0 \in X$ a unique local solution exists. The solution is global by elementary a priori estimates. The claim concerning nonnegativity follows by the maximum principle. □

To prove Theorem 4.1.2 we need a linearized stability analysis of problem (4.0.3), which is conveniently thought of in the abstract form (4.2.2). If so, stationary solutions of problem (4.0.3) satisfy

$$\mathcal{F}(\lambda,\chi,U) \;=\; 0\,. \tag{4.2.3}$$

Clearly, for any $U_1 \equiv (u_1,v_1) \in Y$

$$\mathcal{F}_U(\lambda,\chi,U)U_1 \;=\; \begin{pmatrix} u_1'' + F_u(u,v)u_1 + F_v(u,v)v_1 \\ \lambda v_1'' - \chi[(u'v_1)' + (u_1'v)'] + G_u(u,v)u_1 + G_v(u,v)v_1 \end{pmatrix}\,;$$

hereafter, by $\mathcal{F}_U$, $\mathcal{F}_{\lambda U}$, $\mathcal{F}_{UU}$, $\mathcal{F}_{UUU}$ we denote the Fréchet partial derivatives of $\mathcal{F}$ with respect to its arguments. Observe that $\mathcal{F}_U(\lambda,\chi,U) \in \mathcal{L}(Y,X)$ ($\mathcal{L}(W,Z)$ denoting the space of bounded linear operators from the Banach space W to the Banach space Z).

Let $\bar{U} \equiv (\bar{u},\bar{v})$ be a stationary solution of problem (4.0.4). By the above equality, the linearized operator at $\bar{U}$ of the right-hand side of (4.2.2) is

$$A_{\lambda,\chi} \equiv \mathcal{F}_U(\lambda,\chi,\bar{U}) = \begin{pmatrix} \dfrac{d^2}{dx^2} + F_u & F_v \\ -\chi\bar{v}\dfrac{d^2}{dx^2} + G_u & \lambda\dfrac{d^2}{dx^2} + G_v \end{pmatrix}, \tag{4.2.4}$$

where $F_u \equiv F_u(\bar{u},\bar{v})$, and so on. It is easily seen that the linearized operator $A_{\lambda,\chi}$ has compact resolvent, thus purely point spectrum. Its eigenvalues are the roots $\zeta_n \in \mathbb{C}$ of the equation

$$\left| \begin{array}{cc} \zeta + k_n - F_u & -F_v \\ -\chi\bar{v}k_n - G_u & \zeta + \lambda k_n - G_v \end{array} \right| = 0 \quad \Leftrightarrow \quad \zeta^2 + \phi(\lambda, k_n)\zeta + \psi(\lambda, \chi, k_n) = 0\,, \tag{4.2.5}$$

where $k_n := n^2\pi^2$ ($n \in \mathbb{N} \cup \{0\}$) and the functions ϕ, ψ are defined by (4.1.14), respectively (4.1.9) (here use of equalities (4.1.3) has been made). The corresponding eigenfunctions are

$$\Phi_n \equiv (\varphi_1^n, \varphi_2^n) = \left(a\,cos\left(\sqrt{k_n}x\right), b\,cos\left(\sqrt{k_n}x\right)\right), \tag{4.2.6}$$

with $a, b \in \mathbb{R}$ to be chosen. By the completeness of the trigonometric system it is easily seen that no other eigenfunctions and eigenvalues exist.

In the following we suppose that assumption (A_0) is satisfied. Let us seek conditions on the parameter λ ensuring that some eigenvalue ζ_n of the linearized operator A_λ has positive real part, so that the steady state $(\bar{u}, \bar{v})$ becomes unstable with respect to solutions of the PDE problem (4.0.3). In fact, this amounts to prove Theorem 4.1.2.

Proof of Theorem 4.1.2. Every complex root $\zeta = \zeta_1 + i\zeta_2$ of the equation

$$\zeta^2 + \phi(\lambda, k)\zeta + \psi(\lambda, \chi, k) = 0 \qquad (k \geq 0) \tag{4.2.7}$$

satisfies the system

$$\begin{cases} \zeta_1^2 - \zeta_2^2 + \phi(\lambda, k)\zeta_1 + \psi(\lambda, \chi, k) = 0 \\ \left[2\zeta_1 + \phi(\lambda, k)\right]\zeta_2 = 0\,. \end{cases}$$

Since we seek solutions with $\zeta_1 > 0$, and there holds $\phi(\lambda, k) > 0$ for any $\lambda, k \geq 0$ (see (4.1.14) and recall that $F_u + G_v < 0$ by assumption (A_0)), the second equation gives $\zeta_2 = 0$. Hence solutions of the above system with $\zeta_1 > 0$ exist, if and only if there exist real positive solutions of equation (4.2.7). Since $\phi(\lambda, k) > 0$ for any $\lambda, k \geq 0$, this happens if and only if $\psi(\lambda, \chi, k) < 0$ for some $\lambda > 0$, $\chi \geq 0$ and $k > 0$.

By equalities (4.1.8)-(4.1.9) and (4.1.10) there holds

$$\psi(\lambda, \chi, k) < 0 \;\text{ for some } k > 0 \;\Leftrightarrow\; \begin{cases} F_u\lambda + G_v + \chi\bar{v}F_v > 0 \\ (F_u\lambda + G_v + \chi\bar{v}F_v)^2 + 4F_u(G_u - G_v)\lambda > 0\,. \end{cases}$$

The second inequality of the above system is satisfied if either $\lambda < \lambda_0$, or

$$\lambda > \lambda^{(2)} := \frac{1}{|F_u|}\left(2G_u - G_v + \chi\bar{v}F_v + 2\sqrt{(G_u + \chi\bar{v}F_v)(G_u - G_v)}\right)\,,$$

whereas the first inequality yields

$$\lambda < \lambda^{(1)} := \frac{G_v + \chi\bar{v}F_v}{|F_u|}\,.$$

By assumptions (A_0)-(A_1) there holds $0 < \lambda_0 < \lambda^{(1)} < \lambda^{(2)}$, thus condition (A_2) ensures that the system of inequalities above is satisfied.

Therefore, if χ and λ satisfy conditions (A_1) and (A_2) respectively, there exists $k > 0$ such that $\psi(\lambda, \chi, k) < 0$. On the other hand, there holds $\psi(\lambda, \chi, 0) > 0$ (see (4.1.10)) and $\psi(\lambda, \chi, k) \to \infty$ as $k \to \infty$, since by assumption $\lambda > 0$. Then, by continuity, for every $\chi \in [0, \chi_0)$ and $\lambda \in (0, \lambda_0)$ there exist $0 < k_-(\lambda, \chi) < k_+(\lambda, \chi)$ such that $\psi(\lambda, \chi, k_\pm(\lambda, \chi)) = 0$ and $\psi(\lambda, \chi, k) < 0$ for any $k \in (k_-(\lambda, \chi), k_+(\lambda, \chi))$. Therefore, an eigenvalue of the linearized operator A_λ with positive real part (namely, a real positive solution of equation (4.2.7) with $k = k_n$) exists if and only if inequality (4.1.7) is satisfied for some $n \in \mathbb{N}$. This completes the proof. $\square$

In the remaining part of this section we suppose that $\chi \in [0, \chi_0)$ is fixed, thus we denote by $\mathcal{F}(\lambda, \bar{U}) \equiv \mathcal{F}(\lambda, \chi, \bar{U})$ the operator defined in (4.2.1).

If condition (4.1.15) is satisfied, the roots of equation (4.1.13) are $\zeta_+(k_{n_0}) = 0$, $\zeta_-(k_{n_0}) = -\phi(\lambda, k_{n_0}) < 0$. Then the linearized operator $A_{\lambda_0} \equiv \mathcal{F}_U(\lambda_0, \bar{U})$ is not invertible, since it has an eigenvalue equal to zero. Moreover, if assumptions (A_0)-(A_2) are satisfied, $\bar{U}$ is unstable with respect to problem (4.0.3) for any $\lambda \in (0, \lambda_0)$. This suggests that the point $(\lambda_0, \bar{U})$ is a bifurcation point of equation (4.2.3), as in fact the following proposition shows.

Proposition 4.2.1. *Let the assumptions of Theorem 4.1.3 be satisfied. Then the statements (i)-(ii) of the same theorem hold true.*

To prove Proposition 4.2.1 we need some preliminary remarks. Set $\Phi \equiv (\varphi_1, \varphi_2) \in Y$. Then the eigenvalue equation $A_{\lambda_0}\Phi = \zeta\Phi$ reads (see (4.2.4))

$$\begin{cases} \varphi_1'' + F_u\varphi_1 + F_v\varphi_2 = \zeta\varphi_1 \\ \lambda_0\varphi_2'' - \chi\bar{v}\varphi_1'' + G_u\varphi_1 + G_v\varphi_2 = \zeta\varphi_2 \quad \text{in } \Omega \\ \varphi_1'(0) = \varphi_2'(0) = \varphi_1'(1) = \varphi_2'(1) = 0\,. \end{cases} \tag{4.2.8}$$

It is immediately checked that any vector $E \in Y$,

$$E \equiv (e_1, e_2) := \left(a\cos\left(\sqrt{k_{n_0}}\,x\right), b\cos\left(\sqrt{k_{n_0}}\,x\right)\right) \qquad (x \in \overline{\Omega}) \tag{4.2.9}$$

(see (4.2.6)) is an eigenvector of the linearized operator A_{λ_0} with eigenvalue 0, if k_{n_0} and λ_0 are related by equality (4.1.15) and

$$a \in \mathbb{R} \setminus \{0\}\,, \qquad b := \frac{k_{n_0} - F_u}{F_v}\,a\,. \tag{4.2.10}$$

Observe that in this case the first equality in (4.2.5) reads

$$\begin{vmatrix} k_{n_0} - F_u & -F_v \\ -\chi\bar{v}k_{n_0} - G_u & k_{n_0}\lambda_0 - G_v \end{vmatrix} = 0\,. \tag{4.2.11}$$

It has been already observed that the eigenvalue 0 is simple, thus the kernel $\mathcal{N}(A_{\lambda_0}) \subseteq Y$ of the operator A_{λ_0} coincides with the linear span of the vector E.

Consider also any vector $E^* \in Y$,

$$E^* \equiv (e_1^*, e_2^*) := \left(a^* \cos\left(\sqrt{k_{n_0}}\, x\right), b^* \cos\left(\sqrt{k_{n_0}}\, x\right)\right) \quad (x \in \overline{\Omega}), \tag{4.2.12}$$

with k_{n_0} and λ_0 related by equality (4.1.15) and

$$a^* \in \mathbb{R} \setminus \{0\} \quad b^* := \frac{F_v}{k_{n_0}\lambda_0 - G_v}\, a^* . \tag{4.2.13}$$

It is easily checked that E^* is an eigenvector with eigenvalue 0 of the formal adjoint $A^*_{\lambda_0}$ of A_{λ_0},

$$A^*_{\lambda_0} := \begin{pmatrix} \dfrac{d^2}{dx^2} + F_u & -\chi\bar{v}\dfrac{d^2}{dx^2} + G_u \\ F_v & \lambda_0 \dfrac{d^2}{dx^2} + G_v \end{pmatrix} .$$

It is also easily seen that

$$((E^*, Z)) = 0 \quad \text{for any } \; Z \in \mathcal{R}(A_{\lambda_0}), \tag{4.2.14}$$

where $\mathcal{R}(A_{\lambda_0}) \subseteq X$ denotes the range of the operator A_{λ_0} and $((\cdot,\cdot))$ the scalar product in $L^2(\Omega) \times L^2(\Omega)$, namely

$$((F, G)) := \int_0^1 \{f_1 g_1 + f_2 g_2\}\, dx$$

for any $F \equiv (f_1, f_2), G \equiv (g_1, g_2) \in L^2(\Omega) \times L^2(\Omega)$. In fact, let $Z \equiv (z_1, z_2) \in \mathcal{R}(A_{\lambda_0})$. Then there exists $W \equiv (w_1, w_2) \in Y$ such that $Z = A_{\lambda_0} W$, namely

$$\begin{cases} w_1'' + F_u w_1 + F_v w_2 = z_1 \\ \lambda_0 w_2'' - \chi\bar{v} w_1'' + G_u w_1 + G_v w_2 = z_2 \quad \text{in } \Omega . \end{cases}$$

Since $w_1'(0) = w_2'(0) = w_1'(1) = w_2'(1) = 0$, by the definition of b (see (4.2.10)) and equality (4.2.11) there holds

$$\begin{aligned} ((E^*, Z)) = \frac{1}{2}\Big\{ & a^*\left[\, (F_u - k_{n_0})\, a + F_v b\right] + \\ & + b^*\left[\, (\chi\bar{v}k_{n_0} + G_u)\, a + (G_v - k_{n_0}\lambda_0)\, b\right]\Big\} = 0 . \end{aligned}$$

Further, observe that

$$\begin{aligned} ((E^*, E)) &= (aa^* + bb^*) \int_0^1 \cos^2\left(\sqrt{k_{n_0}}\, x\right) dx = \\ &= \frac{aa^*}{2}\left[1 + \frac{k_{n_0} - F_u}{k_{n_0}\lambda_0 - G_v}\right] . \end{aligned}$$

Then choosing

$$a^* := \frac{2}{a} \frac{k_{n_0}\lambda_0 - G_v}{k_{n_0}(\lambda_0 + 1) - (F_u + G_v)} \tag{4.2.15}$$

we have

$$((E^*, E)) = 1\,. \tag{4.2.16}$$

Without loss of generality, hereafter we suppose

$$a > 0 \tag{4.2.17}$$

and b, b^*, a^* chosen as in (4.2.10), (4.2.13) and (4.2.15), respectively. Observe that by equalities (4.1.3), (4.1.11) and (4.1.15)

$$k_{n_0}\lambda_0 - G_v = -\frac{|F_u|\lambda_0 + G_v + \chi\bar{v}|F_v|}{2} \;<\; 0\,. \tag{4.2.18}$$

Then by assumption (A_0), equality (4.1.3) and inequality (4.2.18) there holds

$$b < 0\,, \quad a^* < 0\,, \quad b^* < 0\,. \tag{4.2.19}$$

Now we can prove Proposition 4.2.1.

Proof of Proposition 4.2.1. Consider the second Fréchet derivative

$$\mathcal{F}_{\lambda U}(\lambda, \bar{U}) = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{d^2}{dx^2} \end{pmatrix}$$

(observe that $\mathcal{F}_{\lambda U}(\lambda, \bar{U}) \in \mathcal{L}(\mathbb{R}, \mathcal{L}(Y, X)) \simeq \mathcal{L}(Y, X)$). By the Lyapunov-Schmidt theorem (*e.g.*, see [1, Theorem 26.13]), the result will follow if we prove that

$$\mathcal{F}_{\lambda U}(\lambda, \bar{U})[\mathcal{N}(A_{\lambda_0})] \not\subseteq \mathcal{R}(A_{\lambda_0})\,, \tag{4.2.20}$$

where $\mathcal{N}(A_{\lambda_0}) \subseteq Y$ denotes the kernel of the operator A_{λ_0} and $\mathcal{F}_{\lambda U}(\lambda, \bar{U})[\mathcal{N}(A_{\lambda_0})]$ its image under the operator $\mathcal{F}_{\lambda U}(\lambda, \bar{U})$.

Let E, E^* be the vectors defined in (4.2.9) and (4.2.12). Clearly, there holds

$$\mathcal{F}_{\lambda U}(\lambda, \bar{U})E = \left(0, -b\, k_{n_0} \cos\left(\sqrt{k_{n_0}}\, x\right)\right)\,,$$

whence by (4.2.19)

$$((E^*, \mathcal{F}_{\lambda U}(\lambda_0, \bar{U})E)) = -bb^*\, k_{n_0} \int_0^1 \cos^2\left(\sqrt{k_{n_0}}\, x\right)\, dx = -\frac{bb^*\, k_{n_0}}{2} < 0\,. \tag{4.2.21}$$

Since $\mathcal{N}(A_{\lambda_0})$ coincides with the linear span of the vector E, by equality (4.2.14) and inequality (4.2.21) we obtain (4.2.20). Then the conclusion follows. $\square$

It is easily seen that for any $U_1 \equiv (u_1, v_1)$ and $U_2 \equiv (u_2, v_2) \in Y$

$$\mathcal{F}_{UU}(\lambda, \bar{U})U_1U_2 = $$
$$= \begin{pmatrix} F_{uu}u_1u_2 + F_{uv}(u_1v_2 + u_2v_1) + F_{vv}v_1v_2 \\ -\chi[(u_2'v_1)' + (u_1'v_2)'] + G_{uu}u_1u_2 + G_{uv}(u_1v_2 + u_2v_1) + G_{vv}v_1v_2 \end{pmatrix},$$

where $F_u \equiv F_u(\bar{u}, \bar{v})$ and so on, whereas

$$\mathcal{F}_{UUU}(\lambda, \bar{U})U_1U_2U_3 = $$
$$= \begin{pmatrix} F_{uuu}u_1u_2u_3 + F_{uuv}\alpha_1(U_1, U_2, U_3) + F_{uvv}\alpha_2(U_1, U_2, U_3) + F_{vvv}v_1v_2v_3 \\ G_{uuu}u_1u_2u_3 + G_{uuv}\alpha_1(U_1, U_2, U_3) + G_{uvv}\alpha_2(U_1, U_2, U_3) + G_{vvv}v_1v_2v_3 \end{pmatrix},$$

where

$$\alpha_1(U_1, U_2, U_3) = u_1u_2v_3 + u_1v_2u_3 + v_1u_2u_3,$$

$$\alpha_2(U_1, U_2, U_3) = u_1v_2v_3 + v_1u_2v_3 + v_1v_2u_3$$

for any $U_1 \equiv (u_1, v_1)$, $U_2 \equiv (u_2, v_2)$ and $U_3 \equiv (u_3, v_3) \in Y$. Observe that $\mathcal{F}_{UU}(\lambda, \bar{U}) \in \mathcal{L}(Y, \mathcal{L}(Y, X)) \simeq \mathcal{L}_2(Y \times Y, X)$ and $\mathcal{F}_{UUU}(\lambda, \bar{U}) \in \mathcal{L}(Y, \mathcal{L}_2(Y \times Y, X)) \simeq \mathcal{L}_3(Y \times Y \times Y, X)$ ($\mathcal{L}_n(W, Z)$, $n \in \mathbb{N}$, denoting the space of bounded multilinear operators from the Banach space $W := \underbrace{Y \times \cdots \times Y}_{\text{n times}}$ to the Banach space X). If $U_1 = U_2 = U_3$ we write $\mathcal{F}_{UU}(\lambda, \bar{U})U_1^2$ and $\mathcal{F}_{UUU}(\lambda, \bar{U})U_1^3$, with obvious meaning of the symbols.

Now we can complete the proof of Theorem 4.1.3.

Proof of Theorem 4.1.3. In view of Proposition 4.2.1, we only have to prove statement *(iii)*.

Let us first prove that the bifurcation is subcritical. Let $\lambda : (-\varepsilon, \varepsilon) \to \mathbb{R}_+$ ($\varepsilon > 0$), $\lambda_0 = \lambda(0)$ be the smooth map which appears in the parametrization of the bifurcation curve $(\lambda(s), U(s)) \subseteq \mathbb{R}_+ \times Y$ (see Remark 4.1.1). By [1, Remark 27.6] and the proof of [1, Proposition 27.7], we have:

$$\lambda'(0) = -\frac{1}{2}\frac{((E^*, \mathcal{F}_{UU}(\lambda_0, \bar{U})E^2))}{((E^*, \mathcal{F}_{\lambda U}(\lambda_0, \bar{U})E))}, \tag{4.2.22}$$

$$\lambda''(0) = -\frac{1}{3}\frac{((E^*, \mathcal{F}_{UUU}(\lambda_0, \bar{U})E^3))}{((E^*, \mathcal{F}_{\lambda U}(\lambda_0, \bar{U})E))}. \tag{4.2.23}$$

Then by [1, Proposition 27.7] and (4.2.20) the claim will follow, if we prove that

$$\lambda'(0) = 0, \quad \lambda''(0) < 0. \tag{4.2.24}$$

Recalling equality (4.2.9) and the definition of the functions F, G (see (4.1.1)), from the above expressions of $\mathcal{F}_{UU}(\lambda,\bar{U})U_1U_2$ and $\mathcal{F}_{UUU}(\lambda,\bar{U})U_1U_2U_3$ we obtain

$$\begin{aligned}
&\mathcal{F}_{UU}(\lambda_0,\bar{U})E^2 = \\
&= \begin{pmatrix} F_{uu}e_1^2 + 2F_{uv}e_1e_2 + F_{vv}e_1^2 \\ \\ -2\chi(e_1'e_2)' + G_{uu}e_1^2 + 2G_{uv}e_1e_2 + G_{vv}e_2^2 \end{pmatrix} = \\
&= 2\begin{pmatrix} -e_1^2 - e_1e_2 \\ \\ -\chi(e_1'e_2)' + \left[\delta - \dfrac{\gamma}{(1+\tau\bar{v})^2}\right]e_1e_2 + \dfrac{\gamma\tau\bar{u}}{(1+\tau\bar{v})^3}e_2^2 \end{pmatrix},
\end{aligned} \tag{4.2.25}$$

respectively

$$\begin{aligned}
\mathcal{F}_{UUU}(\lambda_0,\bar{U})E^3 &= \begin{pmatrix} F_{uuu}e_1^3 + 3F_{uuv}e_1^2e_2 + 3F_{uvv}e_1e_2^2 + F_{vvv}e_2^3 \\ \\ G_{uuu}e_1^3 + 3G_{uuv}e_1^2e_2 + 3G_{uvv}e_1e_2^2 + G_{vvv}e_2^3 \end{pmatrix} = \\
&= \begin{pmatrix} 0 \\ \\ \dfrac{6\gamma\tau}{(1+\tau\bar{v})^3}e_1e_2^2 - \dfrac{6\gamma\tau^2\bar{u}}{(1+\tau\bar{v})^4}e_2^3 \end{pmatrix}.
\end{aligned}$$

It is easily checked that

$$\begin{aligned}
&((E^*,\mathcal{F}_{UU}(\lambda_0,\bar{U})E^2)) = \\
&= 2\left\{-a^2a^* - aa^*b + \left[\delta - \frac{\gamma}{(1+\tau\bar{v})^2}\right]abb^* + \right. \\
&+ \left.\frac{\gamma\tau\bar{u}}{(1+\tau\bar{v})^3}b^2b^*\right\}\int_0^1 \cos^3\left(\sqrt{k_{n_0}}\,x\right)dx - \\
&- 2\chi abb^*k_{n_0}\int_0^1 \sin^2\left(\sqrt{k_{n_0}}x\right)\cos\left(\sqrt{k_{n_0}}x\right)dx = 0\,,
\end{aligned}$$

whence $\lambda'(0) = 0$ by equality (4.2.22). Moreover, there holds

$$((E^*,\mathcal{F}_{UUU}(\lambda_0,\bar{U})E^3)) = \frac{6\gamma\tau b^2b^*}{(1+\tau\bar{v})^3}\left(a - \frac{\tau\bar{u}}{1+\tau\bar{v}}\,b\right)\int_0^1 \cos^4\left(\sqrt{k_{n_0}}\,x\right) < 0$$

(here use of (4.2.17) and (4.2.19) has been made). Then by (4.2.21) and (4.2.23) we obtain that $\lambda''(0) < 0$. This proves (4.2.24), whence the claim follows.

Let us now prove that the stationary bifurcating solutions $U(s) \equiv (u(s), v(s))$ (see (4.1.17)) are asymptotically stable. By [1, Proposition 26.24] there exists a unique continuation $\kappa(s) \in \sigma\big(\mathcal{F}_U(\lambda(s),U(s))\big)$ of the zero eigenvalue of $A_{\lambda_0} \equiv \mathcal{F}_U(\lambda_0,\bar{U})$ along the curve $\{U(s)|s \in (-\varepsilon,\varepsilon)\}$ of bifurcating solutions - namely, there exists a smooth function $(\kappa,\tilde{E}) : (-\varepsilon,\varepsilon) \to \mathbb{R}\times Y$, with $\kappa(0) = 0$ and $\tilde{E}(0) = E$, such that

$$\mathcal{F}_U(\lambda(s),U(s))\tilde{E}(s) = \kappa(s)\tilde{E}(s) \quad \text{for any } \ s \in (-\varepsilon,\varepsilon)\,.$$

By [1, Theorem 27.2] there exists (possibly for some smaller ε) a function $\alpha \in C((-\varepsilon,\varepsilon),\mathbb{R})$ such that

$$\kappa(s) \;=\; \alpha(s)s\lambda'(s) \quad \text{for any } \; s\in(-\varepsilon,\varepsilon)\,; \tag{4.2.26}$$

moreover,

$$\alpha(0) = -((E^*, \mathcal{F}_{\lambda U}(\lambda_0,\bar{U})E))\,. \tag{4.2.27}$$

Since $\lambda'(0)=0$, by (4.2.26) we have

$$\kappa(s) \;=\; \alpha(s)[s^2\lambda''(0)+o(s^2)] \quad \text{as } \; s\to 0\,, \tag{4.2.28}$$

where $o(s^2)$ denotes a term of higher order with respect to s^2. On the other hand, by (4.2.21) and (4.2.27) there holds $\alpha(0) > 0$. Then by continuity of the $\alpha(\cdot)$ and the inequality in (4.2.24), from (4.2.28) we obtain that $\kappa(s) < 0$ for any $|s| \in (0,\varepsilon)$ sufficiently small. Hence the conclusion follows. □

4.3 Nonlocal interaction: proofs

Consider the open subset of the space Y

$$B := \left\{ U \equiv (u,v) \in Y \;\Big|\; \int_0^1 v(x)\,dx \neq 0 \right\}\,,$$

and consider the map $\tilde{\mathcal{F}} : \mathbb{R}_+ \times \overline{\mathbb{R}}_+ \times B \;\to\; X$,

$$\tilde{\mathcal{F}}(\lambda,\chi,U) \;:=\; \begin{pmatrix} u'' + F(u,v) \\ \lambda v'' - \chi(u'v)' + \tilde{G}(u,v) \end{pmatrix}\,,$$

for any $\lambda > 0$, $\chi \geq 0$ and $U \equiv (u,v) \in B$; here

$$\begin{aligned} \tilde{G}(u,v) \;&:=\; -\gamma\,\frac{uv}{1+\tau v} + \delta\,\frac{\langle u,v\rangle}{\langle 1,v\rangle}\,v = \\ &=\; G(u,v) + \delta\left[\frac{\langle u,v\rangle}{\langle 1,v\rangle} - u\right] v\,. \end{aligned}$$

Then problem (4.0.1) can be written in the abstract form

$$\begin{cases} U_t = \tilde{\mathcal{F}}(\lambda,\chi,U) & \text{in } \mathbb{R}_+ \\ U(0) \;= U_0\,. \end{cases} \tag{4.3.1}$$

Observe that

$$\tilde{\mathcal{F}}(\lambda,\chi,U) = \mathcal{F}(\lambda,\chi,U) + \delta \begin{pmatrix} 0 & 0 \\ \mathcal{H}(U) - uv & 0 \end{pmatrix}\,, \tag{4.3.2}$$

where

$$\mathcal{H} : \hat{B} := \left\{ U \equiv (u,v) \in X \,\Big|\, \int_0^1 v(x)\,dx \neq 0 \right\} \;\to\; C(\overline{\Omega})\,, \quad U \to \mathcal{H}(U) := \frac{\langle u, v\rangle}{\langle 1, v\rangle} v\,.$$

By $\mathcal{H}'(U) \in \mathcal{L}(X, C(\overline{\Omega}))$, $\mathcal{H}''(U) \in \mathcal{L}_2(X, C(\overline{\Omega}))$, $\mathcal{H}'''(U) \in \mathcal{L}_3(X, C(\overline{\Omega}))$ we shall denote the first three derivatives of $\mathcal{H}(U)$ evaluated at some $U \in \hat{B}$. The following technical lemma will be used in the sequel (in particular, to obtain the expression (4.1.24) of the Fréchet derivative $\tilde{A}_\lambda \equiv \tilde{\mathcal{F}}_U(\lambda, \chi, \bar{U})$).

Lemma 4.3.1. *Let $U \equiv (u,v) \in \hat{B}$. Then:*
(i) for any $U_1 \equiv (u_1, v_1) \in X$

$$\mathcal{H}'(U)U_1 = \frac{\langle u, v_1\rangle + \langle u_1, v\rangle}{\langle 1, v\rangle} v + \frac{\langle u, v\rangle}{\langle 1, v\rangle} v_1 - \frac{\langle u, v\rangle\langle 1, v_1\rangle}{\langle 1, v\rangle^2} v\,;$$

(ii) for any $U_1 \equiv (u_1, v_1)$ and $U_2 \equiv (u_2, v_2) \in X$

$$\begin{aligned}
\mathcal{H}''(U)U_1U_2 &= \frac{\langle u_2, v_1\rangle + \langle u_1, v_2\rangle}{\langle 1, v\rangle} v + \\
&+ \frac{\langle u, v_1\rangle + \langle u_1, v\rangle}{\langle 1, v\rangle} v_2 + \frac{\langle u, v_2\rangle + \langle u_2, v\rangle}{\langle 1, v\rangle} v_1 - \\
&- \frac{\big[\langle u, v_1\rangle + \langle u_1, v\rangle\big]\langle 1, v_2\rangle + \big[\langle u, v_2\rangle + \langle u_2, v\rangle\big]\langle 1, v_1\rangle}{\langle 1, v\rangle^2} v - \\
&- \frac{\langle u, v\rangle\langle 1, v_2\rangle}{\langle 1, v\rangle^2} v_1 - \frac{\langle u, v\rangle\langle 1, v_1\rangle}{\langle 1, v\rangle^2} v_2 + \frac{2\langle u, v\rangle\langle 1, v_1\rangle\langle 1, v_2\rangle}{\langle 1, v\rangle^3} v\,;
\end{aligned}$$

(iii) for any $U_1 \equiv (u_1, v_1)$, $U_2 \equiv (u_2, v_2)$ and $U_3 \equiv (u_3, v_3) \in X$

$$\begin{aligned}
\mathcal{H}'''(U)U_1U_2U_3 &= \frac{\langle u_2, v_3\rangle + \langle u_3, v_2\rangle}{\langle 1, v\rangle} v_1 + \frac{\langle u_1, v_3\rangle + \langle u_3, v_1\rangle}{\langle 1, v\rangle} v_2 + \\
&+ \frac{\langle u_1, v_2\rangle + \langle u_2, v_1\rangle}{\langle 1, v\rangle} v_3 - \frac{\big[\langle u, v_1\rangle + \langle u_1, v\rangle\big]\langle 1, v_3\rangle}{\langle 1, v\rangle^2} v_2 - \\
&- \frac{\big[\langle u_1, v_3\rangle + \langle u_3, v_1\rangle\big]\langle 1, v_2\rangle + \big[\langle u_2, v_3\rangle + \langle u_3, v_2\rangle\big]\langle 1, v_1\rangle}{\langle 1, v\rangle^2} v - \\
&- \frac{\big[\langle u, v_1\rangle + \langle u_1, v\rangle\big]\langle 1, v_2\rangle + \big[\langle u, v_2\rangle + \langle u_2, v\rangle\big]\langle 1, v_1\rangle}{\langle 1, v\rangle^2} v_3 - \\
&- \frac{\big[\langle u_2, v_1\rangle + \langle u_1, v_2\rangle\big]\langle 1, v_3\rangle}{\langle 1, v\rangle^2} v - \frac{\big[\langle u, v_2\rangle + \langle u_2, v\rangle\big]\langle 1, v_3\rangle}{\langle 1, v\rangle^2} v_1 -
\end{aligned}$$

$$
\begin{aligned}
- & \frac{\big[\langle u, v_3\rangle + \langle u_3, v\rangle\big]\langle 1, v_2\rangle}{\langle 1, v\rangle^2}\, v_1 - \frac{\big[\langle u, v_3\rangle + \langle u_3, v\rangle\big]\langle 1, v_1\rangle}{\langle 1, v\rangle^2}\, v_2 + \\
+ & \frac{2\langle u, v\rangle\langle 1, v_2\rangle\langle 1, v_3\rangle}{\langle 1, v\rangle^3}\, v_1 + \frac{2\langle u, v\rangle\langle 1, v_1\rangle\langle 1, v_3\rangle}{\langle 1, v\rangle^3}\, v_2 + \\
+ & \frac{2\big\{\big[\langle u, v_1\rangle + \langle u_1, v\rangle\big]\langle 1, v_2\rangle + \big[\langle u, v_2\rangle + \langle u_2, v\rangle\big]\langle 1, v_1\rangle\big\}\langle 1, v_3\rangle}{\langle 1, v\rangle^3}\, v + \\
+ & \frac{2\big[\langle u, v_3\rangle + \langle u_3, v\rangle\big]\langle 1, v_1\rangle\langle 1, v_2\rangle}{\langle 1, v\rangle^3}\, v + \frac{2\langle u, v\rangle\langle 1, v_2\rangle\langle 1, v_1\rangle}{\langle 1, v\rangle^3} v_3 - \\
- & \frac{6\langle u, v\rangle\langle 1, v_1\rangle\langle 1, v_2\rangle\langle 1, v_3\rangle}{\langle 1, v\rangle^4}\, v\,.
\end{aligned}
$$

Proof. We only prove claim (i); the lengthy proof of (ii)-(iii) is similar, thus we omit it. For any $U \equiv (u, v) \in \hat{B}$, $U_1 \equiv (u_1, v_1) \in X$ and $\varepsilon > 0$ sufficiently small there holds $U + \varepsilon U_1 \in \hat{B}$. Then, denoting by $o(\varepsilon)$ any term of higher order with respect to ε, we have that

$$
\begin{aligned}
\mathcal{H}(U + \varepsilon U_1) &= \frac{\langle u + \varepsilon u_1, v + \varepsilon v_1\rangle}{\langle 1, v + \varepsilon v_1\rangle}\,(v + \varepsilon v_1) = \\
&= \frac{\langle u, v\rangle + \varepsilon[\langle u, v_1\rangle + \langle u_1, v\rangle] + o(\varepsilon)}{\langle 1, v\rangle\left(1 + \varepsilon\dfrac{\langle 1, v_1\rangle}{\langle 1, v\rangle}\right)}\,(v + \varepsilon v_1) = \\
&= \frac{\langle u, v\rangle + \varepsilon[\langle u, v_1\rangle + \langle u_1, v\rangle] + o(\varepsilon)}{\langle 1, v\rangle}\left(1 - \varepsilon\,\frac{\langle 1, v_1\rangle}{\langle 1, v\rangle} + o(\varepsilon)\right)(v + \varepsilon v_1) = \\
&= \frac{\langle u, v\rangle + \varepsilon[\langle u, v_1\rangle + \langle u_1, v\rangle] + o(\varepsilon)}{\langle 1, v\rangle}\left(v + \varepsilon v_1 - \varepsilon\,\frac{\langle 1, v_1\rangle}{\langle 1, v\rangle}\, v + o(\varepsilon)\right) \\
&= \mathcal{H}(u, v) + \frac{\varepsilon[\langle u, v_1\rangle + \langle u_1, v\rangle]}{\langle 1, v\rangle}\, v + \varepsilon\frac{\langle u, v\rangle}{\langle 1, v\rangle}\, v_1 - \varepsilon\frac{\langle u, v\rangle\langle 1, v_1\rangle}{\langle 1, v\rangle^2}\, v + o(\varepsilon)\,.
\end{aligned}
$$

Hence the claim follows. □

Now we can show that the linearized operator $\tilde{A}_\lambda \equiv \tilde{\mathcal{F}}_U(\lambda, \chi, \bar{U})$ at the constant stationary solution $\bar{U}$ has the expression given by (4.1.24). In fact, applying the Fréchet derivative of the operator-valued matrix in equality (4.3.2) to any $U_1 \equiv (u_1, v_1) \in X$ we get

$$
\begin{pmatrix} 0 & 0 \\ \mathcal{H}'(\bar{U})U_1 - \bar{u}v_1 - \bar{v}u_1 & 0 \end{pmatrix}. \tag{4.3.3}
$$

Since $\bar{u}$ and $\bar{v}$ are constant, there holds

$$
\langle 1, \bar{u}\rangle = \bar{u}\,, \quad \langle 1, \bar{v}\rangle = \bar{v}\,.
$$

Then by Lemma 4.3.1 we plainly obtain

$$
\begin{aligned}
\mathcal{H}'(\bar{U})U_1 &= \frac{\langle \bar{u}, v_1\rangle + \langle u_1, \bar{v}\rangle}{\langle 1, \bar{v}\rangle}\,\bar{v} + \frac{\langle \bar{u}, \bar{v}\rangle}{\langle 1, \bar{v}\rangle}\, v_1 - \frac{\langle \bar{u}, \bar{v}\rangle\langle 1, v_1\rangle}{\langle 1, \bar{v}\rangle^2}\,\bar{v} = \\
&= \bar{u}\langle 1, v_1\rangle + \bar{v}\langle 1, u_1\rangle + \bar{u}v_1 - \bar{u}\langle 1, v_1\rangle = \\
&= \bar{v}\langle 1, u_1\rangle + \bar{u}v_1\,,
\end{aligned}
$$

thus

$$\mathcal{H}'(\bar{U})U_1 - \bar{u}v_1 - \bar{v}u_1 = \bar{v}\left[\langle 1, u_1\rangle - u_1\right] .$$

By (4.3.2) and the above equality we obtain

$$\tilde{A}_\lambda = A_\lambda + \delta \begin{pmatrix} 0 & 0 \\ \bar{v}\left[\langle 1, \cdot\rangle - 1\right] & 0 \end{pmatrix}, \tag{4.3.4}$$

whence equality (4.1.24) follows. In particular, equality (4.3.4) shows that the operator $\tilde{A}_\lambda$ has compact resolvent (since this holds for A_λ), thus its spectrum consists of eigenvalues.

In view of (4.1.24), the eigenvalue equation $\tilde{A}_\lambda\Phi = \zeta\Phi$ $(\Phi \equiv (\varphi_1, \varphi_2) \in Y)$ reads

$$\begin{cases} \varphi_1'' + F_u\varphi_1 + F_v\varphi_2 = \zeta\varphi_1 \\ \lambda\varphi_2'' - \chi\bar{v}\varphi_1'' + G_u\varphi_1 + \delta\bar{v}\left[\langle 1, \varphi_1\rangle - \varphi_1\right] + G_v\varphi_2 = \zeta\varphi_2 \quad \text{in } \Omega \\ \varphi_1'(0) = \varphi_2'(0) = \varphi_1'(1) = \varphi_2'(1) = 0\,. \end{cases} \tag{4.3.5}$$

As for system (4.2.8), choose

$$\Phi_n \equiv (\varphi_1^n, \varphi_2^n) = \left(a\,cos\left(\sqrt{k_n}x\right), b\,cos\left(\sqrt{k_n}x\right)\right),$$

with $k_n := n^2\pi^2$ $(n \in \mathbb{N} \cup \{0\})$ and $a, b \in \mathbb{R}$ to be fixed, as trial functions. Observe that

$$\langle 1, \varphi_1^0\rangle = a \quad \Rightarrow \quad \left[\langle 1, \varphi_1^0\rangle - \varphi_1^0\right] = 0\,,$$

$$\langle 1, \varphi_1^n\rangle = 0 \quad \Rightarrow \quad \left[\langle 1, \varphi_1^n\rangle - \varphi_1^n\right] = -\varphi_1^n \quad \text{for any } n \in \mathbb{N}\,. \tag{4.3.6}$$

Hence $\Phi_0 \equiv (a, b)$ is an eigenfunction of $\tilde{A}_\lambda$ if and only if ζ is a root of the equation

$$\begin{vmatrix} \zeta - F_u & -F_v \\ -G_u & \zeta - G_v \end{vmatrix} = 0\,, \tag{4.3.7}$$

namely an eigenvalue of the linearized operator (4.1.6) without space dependence. By assumption (A_0) both eigenvalues of this operator have negative real part. Therefore, to have Turing destabilization of the stationary solution $\bar{U}$ we must consider eigenvalues of system (4.3.5) with $n \in \mathbb{N}$. By (4.3.6), these are the roots of the equation

$$\begin{vmatrix} \zeta + k_n - F_u & -F_v \\ -\chi\bar{v}k_n - G_u + \delta\bar{v} & \zeta + \lambda k_n - G_v \end{vmatrix} = 0 \quad \Leftrightarrow \quad \zeta^2 + \phi(\lambda, k_n)\zeta + \tilde{\psi}(\lambda, \chi, k_n) = 0\,, \tag{4.3.8}$$

where $\phi(\lambda, k)$ and $\tilde{\psi}(\lambda, \chi, k)$ are defined by (4.1.14) and (4.1.20), respectively.

Let us now prove Theorem 4.1.4.

Proof of Theorem 4.1.4. As in the proof of Theorem 4.1.2, by (4.3.8) a necessary condition for the Turing destabilization of $\bar{U}$ is the existence of real positive solutions of the equation

$$\zeta^2 + \phi(\lambda, k)\zeta + \tilde{\psi}(\lambda, \chi, k) = 0 \qquad (k > 0)\,. \tag{4.3.9}$$

Since $\phi(\lambda, k) > 0$, such solutions exist if and only if $\tilde{\psi}(\lambda, \chi, k) < 0$, namely if and only if $0 < k < \tilde{k}(\lambda, \chi)$ (see Figure 4.1.5). Therefore, a positive eigenvalue of system (4.3.5) exists if and only if $0 < k_n < \tilde{k}(\lambda, \chi)$ for some $n \in \mathbb{N}$. Since $0 < k_1 < \cdots < k_n < \ldots$, for any $\chi \geq 0$ this happens if and only if

$$0 < k_1 < \tilde{k}(\lambda, \chi) \quad \Longleftrightarrow \quad \lambda \in (0, \lambda_1)\,.$$

Then the conclusion follows. □

The following analogue of Proposition 4.2.1 holds true.

Proposition 4.3.2. *Let the assumptions of Theorem 4.1.5 be satisfied. Then* $(\lambda_1, \bar{U})$ *is a bifurcation point of nonconstant stationary solutions of problem* (4.0.1).

The proof of the above proposition is almost verbatim the same of Proposition 4.2.1 (observe that $\mathcal{F}_{\lambda U}(\lambda, \bar{U}) = \tilde{\mathcal{F}}_{\lambda U}(\lambda, \bar{U})$ by equality (4.3.2)), thus we omit it. Let us only mention for future reference that in the present case the vectors E, E^* are replaced by the vectors $D, D^* \in Y$,

$$D \equiv (d_1, d_2) := \left(c \cos\left(\sqrt{k_1}\, x\right), d \cos\left(\sqrt{k_1}\, x\right)\right) \qquad (x \in \overline{\Omega}) \tag{4.3.10}$$

with

$$c \in \mathbb{R} \setminus \{0\}\,, \qquad d := \frac{k_1 - F_u}{F_v}\, c\,, \tag{4.3.11}$$

and

$$D^* \equiv (d_1^*, d_2^*) := \left(c^* \cos\left(\sqrt{k_1}\, x\right), d^* \cos\left(\sqrt{k_1}\, x\right)\right) \quad (x \in \overline{\Omega})\,, \tag{4.3.12}$$

with

$$c^* := \frac{2}{c} \frac{k_1\lambda_1 - G_v}{k_1(\lambda_1 + 1) - (F_u + G_v)} \quad d^* := \frac{F_v}{k_1\lambda_1 - G_v}\, c^*\,. \tag{4.3.13}$$

By the above choice in (4.3.11) and (4.3.13), there holds $((D^*, D)) = 1$. Without loss of generality, we assume that

$$c > 0\,. \tag{4.3.14}$$

Then, by equalities (4.1.3) and assumption (A_0), from (4.3.11) and (4.3.13) we get

$$d < 0\,, \quad d^* = \frac{2}{c} \frac{F_v}{k_1(\lambda_1 + 1) - (F_u + G_v)} < 0\,. \tag{4.3.15}$$

Now we can prove Theorem 4.1.5.

Proof of Theorem 4.1.5. By Proposition 4.3.2, we only have to prove that the bifurcation is subcritical, and the bifurcating nonconstant stationary solutions are asymptotically stable.

By the same notations used in the proof of Theorem 4.1.3, now we have (see (4.2.22)-(4.2.23))

$$\lambda'(0) = -\frac{1}{2}\frac{((D^*, \tilde{\mathcal{F}}_{UU}(\lambda_1, \bar{U})D^2))}{((D^*, \tilde{\mathcal{F}}_{\lambda U}(\lambda_1, \bar{U})D))}, \tag{4.3.16}$$

$$\lambda''(0) = -\frac{1}{3}\frac{((D^*, \tilde{\mathcal{F}}_{UUU}(\lambda_1, \bar{U})D^3))}{((D^*, \tilde{\mathcal{F}}_{\lambda U}(\lambda_1, \bar{U})D))}. \tag{4.3.17}$$

By equality (4.3.2) there holds

$$\tilde{\mathcal{F}}_{UU}(\lambda_1, \bar{U})D^2 = \mathcal{F}_{UU}(\lambda_1, \bar{U})D^2 + \delta \begin{pmatrix} 0 & 0 \\ \mathcal{H}''(\bar{U})D^2 - 2d_1d_2 & 0 \end{pmatrix}. \tag{4.3.18}$$

Since $\bar{U}$ is constant and

$$\langle 1, d_1 \rangle = c\int_0^1 \cos\left(\sqrt{k_1}\,x\right) dx = 0 = \langle 1, d_2 \rangle$$

(see (4.3.10)), from Lemma 4.3.1-(*ii*) plainly we get

$$\mathcal{H}''(\bar{U})D^2 = 2\int_0^1 d_1(x)d_2(x)\,dx\,.$$

Then from (4.3.18) and the above equality we obtain

$$\tilde{\mathcal{F}}_{UU}(\lambda_1, \bar{U})D^2 = 2\begin{pmatrix} -d_1^2 - d_1d_2 \\ -\chi(d_2d_1')' + \delta\int_0^1 d_1d_2\,dx - \frac{\gamma}{(1+\tau\bar{v})^2}\,d_1d_2 + \frac{\gamma\tau\bar{u}}{(1+\tau\bar{v})^3}d_2^2 \end{pmatrix}$$

(see (4.2.25)). Then there holds

$$\begin{aligned} &((D^*, \tilde{\mathcal{F}}_{UU}(\lambda_1, \bar{U})D^2)) = \\ =\;& 2\left\{-c^2c^* - cc^*d - \frac{\gamma}{(1+\tau\bar{v})^2}\,cdd^* + \frac{\gamma\tau\bar{u}}{(1+\tau\bar{v})^3}\,d^2d^*\right\}\int_0^1 \cos^3\left(\sqrt{k_1}\,x\right) dx\; + \\ +\;& 2\delta\,cdd^*\int_0^1 \cos^2\left(\sqrt{k_1}\,x\right) dx \int_0^1 \cos\left(\sqrt{k_1}\,x\right) dx\; - \\ -\;& 2k_1\chi cdd^*\int_0^1 \sin^2\left(\sqrt{k_1}x\right)\cos\left(\sqrt{k_1}x\right) dx = 0\,, \end{aligned}$$

thus $\lambda'(0) = 0$ by (4.3.16).

On the other hand, from Lemma 4.3.1-(*iii*) we easily obtain

$$\mathcal{H}'''(\bar{U})D^3 = \frac{6}{\bar{v}}\,cd^2\left(\int_0^1 d_1(x)d_2(x)\,dx\right)d_2\,.$$

Then by (4.3.18) and the above equality we have

$$\tilde{\mathcal{F}}_{UUU}(\lambda_1, \bar{U})D^3 = \left(\begin{array}{c} 0 \\ \\ \dfrac{6\gamma\tau}{(1+\tau\bar{v})^3} d_1 d_2^2 - \dfrac{6\gamma\tau^2\bar{u}}{(1+\tau\bar{v})^4} d_2^3 + \dfrac{6\delta}{\bar{v}}\, c d^2 \left(\displaystyle\int_0^1 d_1 d_2 \, dx \right) d_2 \end{array} \right),$$

whence

$$\begin{aligned} ((D^*, \tilde{\mathcal{F}}_{UUU}(\lambda_1, \bar{U})D^3)) \quad &= \quad \frac{6\gamma\tau d^2 d^*}{(1+\tau\bar{v})^3} \left(c - \frac{\tau\bar{u}}{1+\tau\bar{v}}\, d \right) \int_0^1 \cos^4\left(\sqrt{k_1}\, x\right) + \\ &+ \quad \frac{6\delta}{\bar{v}}\, c d^2 d^* \left(\int_0^1 \cos^2\left(\sqrt{k_1}\, x\right) \right)^2 < 0 \end{aligned}$$

(here use of (4.3.14)-(4.3.15) has been made). Moreover, arguing as for (4.2.21) we have

$$((D^*, \tilde{\mathcal{F}}_{\lambda U}(\lambda_1, \bar{U})D)) = -dd^* k_1 \int_0^1 \cos^2\left(\sqrt{k_1}\, x\right) dx < 0\,, \tag{4.3.19}$$

thus $\lambda''(0) < 0$ by equality (4.3.17). Then the same argument used in the proof of Proposition 4.2.1 proves that the bifurcation is subcritical.

Finally, replacing equality (4.2.27) by

$$\alpha(0) = -((D^*, \tilde{\mathcal{F}}_{\lambda U}(\lambda_1, \bar{U})D)) \tag{4.3.20}$$

and inequality (4.2.21) by (4.3.19), the above calculations and the same arguments used in the proof of Theorem 4.1.3 prove that the stationary bifurcating solutions are asymptotically stable. This completes the proof. □

Appendix

A Some auxiliary lemmas

For the reader convenience we formulate here some auxiliary lemmas used throughout this work.

Lemma A.1. (*see [14]*) *Let $1 < p < +\infty$. There exist positive constants c_p, C_p such that for every $\xi, \eta \in \mathbb{R}^n$*

$$c_p N_p(\xi,\eta) \leq (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \leq C_p N_p(\xi,\eta),$$

where

$$N_p(\xi,\eta) = \{|\xi| + |\eta|\}^{p-2}|\xi - \eta|^2,$$

a dot denotes the Euclidean product in R^n.

Lemma A.2. *Let $\xi, \eta \in \mathbb{R}^n$, $p \geq 2$. Then there exists a positive constant C_p such that*

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \geq C_p|\xi - \eta|^p.$$

Proof. If $\xi = \eta$ then statement of the lemma holds true. Thus we can assume that $\xi \neq \eta$. From Lemma above for $p \geq 2$ we get

$$\begin{aligned}(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \geq c_p N_p(\xi,\eta) &= c_p|\xi - \eta|^p \frac{\big(|\xi| + |\eta|\big)^{p-2}}{|\xi - \eta|^{p-2}} \\ &\geq c_p|\xi - \eta|^p \frac{\big(|\xi| + |\eta|\big)^{p-2}}{\big(|\xi| + |\eta|\big)^{p-2}} = C_p|\xi - \eta|^p.\end{aligned}$$

□

Lemma A.3. *Let a, b be non negative numbers. Then*

$$|a^p - b^p| \leq p|a - b|\{a + b\}^{p-1}.$$

Proof. We can suppose $a > b$. Then

$$a^p - b^p = \int_0^1 \frac{d}{dt}|b + t(a-b)|^p dt \leq p\int_0^1 |b + t(a-b)|^{p-1}\frac{b + t(a-b)}{b + t(a-b)}(a-b)dt$$

$$\leq p(a-b)\int_0^1 |ta + (1-t)b|^{p-1}dt \leq p(a-b)\int_0^1 \{|a| + |b|\}^{p-1}dt = p|a-b|\{a+b\}^{p-1}.$$

□

Lemma A.4. *Let $p \geq 2$, $a, b \in \mathbb{R}$. Then*

$$\int_0^1 (1-s)|a+sb|^{p-2}|b|^2 ds \geq \frac{1}{8(18)^{\frac{p}{2}}}|b|^p.$$

Proof.

(i) Let us first assume that $|a| \geq |b|$. Then we have that

$$|a+sb| \geq |a| - s|b| \geq |b| - s|b| = (1-s)|b|, \quad s \in [0,1].$$

Consider now

$$\int_0^1 (1-s)|a+sb|^{p-2}|b|^2 ds \geq \int_0^1 (1-s)^{p-1}|b|^p ds = \frac{|b|^p}{p}$$

and the statement of the lemma holds.

(ii) Let now $|a| < |b|$. Then we see

$$|a+sb| \leq |a| + s|b| < (1+s)|b| \leq 2|b| \quad s \in [0,1].$$

Hence,

$$\int_0^1 (1-s)|a+sb|^{p-2}|b|^2 ds = \int_0^1 (1-s)\frac{|a+sb|^p}{|a+sb|^2}|b|^2 ds \geq \frac{1}{4}\int_0^1 (1-s)|a+sb|^p ds.$$

Since $\int_0^1 2(1-s)ds = 1$ and $X \to X^{\frac{p}{2}}$ is convex by Jensens's inequality we get

$$\int_0^1 (1-s)(|a+sb|^2)^{\frac{p}{2}} ds \geq \frac{1}{2}\left(\int_0^1 2(1-s)(|a|^2 + 2sab + s^2|b|^2)ds\right)^{\frac{p}{2}}$$
$$= \frac{1}{2}\Big(|a|^2 + \frac{2}{3}ab + \frac{1}{6}|b|^2\Big)^{\frac{p}{2}} \geq \frac{1}{2}\Big(|a|^2 - \frac{2}{3}|a||b| + \frac{1}{6}|b|^2\Big)^{\frac{p}{2}}.$$

Using the Young inequality $ab \leq \frac{3a^2}{2} + \frac{b^2}{6}$ and combining the two inequalities above we obtain the statement of the lemma.

□

Lemma A.5. *Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function with $g(x) > 0$ $\forall x > 0$ or such that*

$$\forall \alpha > 0 \text{ small}, \qquad \sup_{[\alpha,2\alpha]} g = C_\alpha > 0. \tag{A.21}$$

Let y, h be nonnegative functions, y continuous such that

$$\int_0^{+\infty} y(s)ds, \int_0^{+\infty} h(s)ds < +\infty,$$

$$y(t) - y(s) \leq \int_s^t (g(y(\xi)) + h(\xi))d\xi, \ \forall s < t. \tag{A.22}$$

Then it holds that

$$\lim_{t\to+\infty} y(t) = 0.$$

Proof. From the condition $\int_0^{+\infty} y(s)ds$ we have that $\liminf_{t\to+\infty} y(t) = 0$. Suppose $\limsup_{t\to+\infty} y(t) > 0$ and choose α such that $\limsup_{t\to+\infty} y(t) > 2\alpha$. By the mean value theorem one can find a sequence of disjoint intervals (t_n, t'_n), $t_n \to +\infty$ such that

$$y(t_n) = \alpha \le y(t) \le 2\alpha = y(t'_n) \quad \forall t \in (t_n, t'_n).$$

Then from the last inequality of (A.22) and (A.21) it holds that

$$\alpha = y(t'_n) - y(t_n) \le \int_{t_n}^{t'_n} g(y(s))ds + \int_{t_n}^{t'_n} h(s)ds \le C_\alpha(t'_n - t_n) + \int_{t_n}^{t'_n} h(s)ds.$$

For $n \ge n_0$ large enough, by (A.22), $\int_{t_n}^{t'_n} h(s)ds \le \frac{\alpha}{2}$ and from above we get

$$t'_n - t_n \ge \frac{\alpha}{2C_\alpha}.$$

It follows that

$$\int_{t_{n_0}}^{+\infty} y(s)ds \ge \sum_{n\ge n_0} \int_{t_n}^{t'_n} y(s)ds \ge \sum_{n\ge n_0} \frac{\alpha^2}{2C_\alpha} = +\infty$$

and a contradiction. □

Lemma A.6. *Let $y(t) \ge 0$ solution to*

$$y'(t) + \beta y^\alpha(t) \le \varepsilon, \ t \ge 0 \tag{A.23}$$

with $\alpha, \beta > 0$, $\varepsilon \ge 0$. Then

$$y(t) \le \max\left\{ y(0), \left(\frac{\varepsilon}{\beta}\right)^{\frac{1}{\alpha}} \right\} \quad \forall t \ge 0. \tag{A.24}$$

Moreover, if $\varepsilon = \varepsilon(t)$ and

$$\lim_{t\to\infty} \varepsilon(t) = 0, \tag{A.25}$$

then

$$\lim_{t\to\infty} y(t) = 0. \tag{A.26}$$

Proof. Denote by $y_0 = y(0)$ and by $y_* = \left(\frac{\varepsilon}{\beta}\right)^{\frac{1}{\alpha}}$. Suppose first that $\max\{y_0, y_*\} = y_*$. We want to prove (A.24) holds true. Let us assume the contrary, i.e. we can define

$$t_* = \inf\{t > 0 : y(t) > y_*\} < +\infty,$$

which implies

$$\forall t \in [0, t_*) \; y(t) \le y_*,$$

$$y(t_*) = y_*,$$

$$\exists \delta > 0 \; y(t_* + \delta) > y_*. \tag{A.27}$$

Hence, using also (A.23), we get

$$y'(t_*) \le \varepsilon - \beta y^\alpha(t_*) = \varepsilon - \beta y_*^\alpha = 0.$$

Since y' is nonincreasing at point t_* we can deduce that $y(t_* + \delta) \le y(t_*)$. But from (A.27) we obtain

$$y_* < y(t_* + \delta) \le y(t_*) = y_*.$$

And we came to a contradiction.
If now $\max\{y_0, y_*\} = y_0$. Define

$$t_0 = \inf\{t > 0 : y(t) > y_0 \ge y_*\} < +\infty,$$

i.e.

$$\forall t \in [0, t_0) \; y(t) \le y_0,$$

$$y(t_0 + \delta) > y_0, \;\; \delta > 0.$$

Then we have

$$0 < y(t_0 + \delta) - y_0 \le y(t_0 + \delta) - y(t_0) = \int_{t_0}^{t_0+\delta} y'(s)ds \le \int_{t_0}^{t_0+\delta} (\varepsilon - \beta y^\alpha)(s)ds \le 0,$$

which is a contradiction.

Now we prove the second part of the lemma. Suppose that (A.26) fails. Then one can find $\delta > 0$ and a sequence t_n, $n = 1, 2, \ldots$ such that

$$y(t_n) \ge \delta, \quad t_n \to +\infty \text{ when } n \to +\infty. \tag{A.28}$$

By (A.25) there exists t_0 such that

$$\varepsilon(t) < \beta\delta^\alpha \quad \forall t \ge t_0.$$

We claim that y is decreasing on $(t_0, +\infty)$. Indeed, let t_n be an arbitrary point such that $t_n > t_0$. One has

$$y'(t_n) \le \varepsilon(t_n) - \beta y(t_n)^\alpha \le \varepsilon(t_n) - \beta\delta^\alpha < 0.$$

Thus y' is negative on the left of t_n - i.e. $y \searrow$ on the left of t_n.

Define

$$\sigma = \inf\{t_0 \le t < t_n \mid y \text{ is decreasing on } (t, t_n)\}.$$

If $\sigma > t_0$ one has $y(\sigma) \geq \delta$ and thus

$$y'(\sigma) \leq \varepsilon(\sigma) - \beta\delta^\alpha < 0,$$

which contradicts the definition of σ. Thus $y \searrow$ on (t_0, t_n) and since t_n is arbitrary we have $y \searrow$ on $(t_0, +\infty)$. Let us define l the limit of y - i.e.

$$\lim_{t \to +\infty} y(t) = l.$$

Suppose $l > 0$, then since $y(t) \geq l$ for t large enough - i.e. $t \geq t_0'$ one has

$$y'(t) \leq \varepsilon(t) - \beta l^\alpha < -\frac{\beta l^\alpha}{2}.$$

Integrating we get

$$y(t) - y(t_0') \leq -\frac{\beta l^\alpha}{2}(t - t_0).$$

But then for t large $y(t) < 0$, which is not possible. We have then $l = 0$, which contradicts (A.28). Thus (A.28) cannot occur and the proof is complete. □

Remark A.1. For $\alpha > 1$ and y absolutely continuous function one can show that [39, Lemma 5.1]

$$y(t) \leq \left(\frac{\varepsilon}{\beta}\right)^{\frac{1}{\alpha}} + \left(\frac{\beta(\alpha - 1)}{t}\right)^{\frac{1}{\alpha - 1}}.$$

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